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The Grain Density and the Process of Track Formation in Nuclear Emulsions. (*)

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PART I.

GRAIN DENSITY AND DEVELOPING FACTOR

Summary. — A critical survey of various methods employed for grain density measurements is given, and a new scheme for the process of track formation is proposed. According to this scheme, which takes into account a process of induced development, grain density for a given emulsion, appears to depend on two main factors, namely, the probability that a grain be impressioned, which is obviously directly related to the energy loss, and what we call the « developing factor » K, which is characteristic of the degree of development of the plate. Both these factors can be easily determined experimentally. A series of measurements alongs tracks due to particles of different energy, appears to confirm the validity of our scheme.

One of the standard experimental data used for the identification of the type of particle that has produced a track in a nuclear emulsion, is the grain density, i.e., the number of developed grains per unit length. Though the process which renders a grain developable is undoubtedly strictly related to

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the energy loss of the traversing particle, as it depends solely on the number of ions produced throughout the grain itself, no precise relation between energy loss and grain density has yet been established with fair certainty. As long as the dE/dR values are less than four times the minimum value, it is generally assumed that the relationship is one of direct proportionality, but above this limiting value, no such simple law can be expected to hold: the track then presents a saturation effect, which depends on the type of emulsion, and on the degree of development.

P. DEMERS ([1]), by introducing some simplifying assumptions, has given a formula which relates the probability π that a grain be impressioned, to the energy loss. But though Demers' formula presents a notable interest, also from a practical point of view, no reliable experimental confirmation of its validity has up to now been possible, as there is no way by which to determinate π. Grain density, as it is actually measured, represents the number of black grains, per unit length of track, observed in the processed plate; it therefore includes, not only the grains actually rendered developable by the traversing particle, but also grains whose blackening is due merely to the process of development, or to the proximity of one or more of the impressioned grains ([2]). From an experimental point of view, we are thus confronted with the practical impossibility of taking into account, quantitatively, the degree of development of the plate.

In the present paper, a critical survey of the various methods employed to determine the grain density is followed by a new scheme for the process of track formation, which enables one to evaluate the probability π, and also a certain coefficient K, characteristic of the degree of development of the emulsion. In Part II, shortly to follow, we shall discuss Demers' relation, by comparing theoretical results with experimental π values; these were deduced from a series of measurements made on tracks due to particles, whose mass and energy could be considered known « a priori ».

1. – Methods employed in grain density measurements.

The method generally used in grain density measurements, consists in counting, under a microscope, the number of black grains for every 100 μ of track. Since the grains usually present themselves in unresolved clusters, which for brevity's sake we shall call grain groups, the observer must evaluate the number of grains pertaining to each clot. But, by so doing, he introduces in

([1]) P. DEMERS: *Can. Journ. Res.*, A **25**, 233 (1947).
([2]) L. P. CLERC: *Structure et Propriété des couches photographiques* (1949), p. 54.

the final result a subjective factor, by no means easy to determine, which not only renders such grain counts scarcely significant, but also bars the possibility for any direct comparison between the results obtained by different observers. L. JAUNEAU and F. HUG-BOUSSER [3] have studied this personal factor, and actually found that, while observers, not specifically trained, count in different ways in successive tests, even well-trained observers introduce in the final result a « personal dispersion », which is far from negligible.

Several tests carried out in this Laboratory have confirmed the conclusions of the above Authors.

The results obtained by measuring the length of each group, and then deducing the number of grains it contains by dividing its length by the mean diameter of a single grain, would appear to be much more significant.

This method, however, is extremely laborious, and, in practice, in all those cases when it is necessary to determine the grain density along a considerable number of tracks, the amount of time required for the actual measurements represents quite a serious drawback.

In view of speeding up such measurements, MORRISH [4] suggests that grain counting might be replaced by group counting, at least for those tracks whose ionization is close to the minimum value, assuming that, with this limitation, the total number of grains be can considered proportional to the total number of groups. The « dispersion » would naturally also be reduced; but, on the other hand, the assumed proportionality is not justifiable « a priori », and even were it legitimate for grain densities close to the minimum value, it is certainly no longer valid in the case of particles whose ionization exceeds this minimum by a factor of only two or three.

The above methods of grain density measurements are based on the hypothesis that a track is formed of a series of grains, which, if traversed by a charged particle, will appear black in the developed plate. If we call p the probability that a grain will be developed, and N the total number of grains along a given length L_t, the number of black grains contained in this length will obviously be

$$n = pN \,, \tag{1}$$

i.e., there is a direct proportionality between grain density and probability p.

If d is the average diameter of one grain, the total length of the grain groups will be

$$L_n = pNd \quad \text{and also} \quad L_t = Nd \,. \tag{2}$$

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(3) L. JAUNEAU and F. HUG-BOUSSER: *Journ. de Phys.*, **13**, 465 (1952).

(4) A. H. MORRISH: *Phil. Mag.*, **43**, 533 (1952).

Following this scheme, we readily find that the total number of groups, n_g, and the number of groups containing i grains, n_i, are given by

$$n_g = p(1-p)\,N\,, \tag{3}$$

$$n_i = p^i(1-p)^2 N\,. \tag{4}$$

From (1) and (3), it is evident that there will be a direct proportionality between number of groups and number of grains only as long as the values are very small, that is to say, only if $1-p \sim 1$.

We have checked the validity of this scheme by a number of measurements along ~ 50 tracks, (practically parallel to the plane of the emulsion) chosen at random on two plates (A) and (B), each of which had been exposed to a beam of artificial mesons, (of ~ 70 and ~ 100 MeV). The tracks that were investigated were chosen at random among those closely collimated with the incident meson beam.

The length of these tracks was of a few thousand μ, and the variation of the energy loss along each individual track could be considered negligible. For each one, we determined the total length, the total number of groups,

TABLE I. *Measurements on track of Plate A.*

$L_t = 6500\ \mu$ $\quad L' = 0.885\ \mu$ $\quad L_n = 1\,443\ \mu$
$d = 0{,}76\ \mu$ $\quad n_g = 1\,553$ $\quad p = 0.222$

n	n_i exp.	n_i calc. by (4)	n_i exp. values corr. by (6)
n_1	1 186	1 192	1 185
n_2	261	265	271
n_3	65	58	59
n_4	15	13	12
n_5	6	4	5

and the length of every single group: as an example, Table I gives the experimental values found for a track of 6 500 μ. From these data, we obtain immediately $p = L_n/L_t$; however, to deduce the grain density, it is necessary to know the values of either d or N. The d value might be obtained directly by measuring the length of groups which appear to consist of a single grain; but it is preferable to deduce it indirectly from other experimental data, by

first evaluating the average length of the groups, L'. We have

$$(5) \qquad L' = \frac{L_n}{n_g} = \frac{pNd}{p(1-p)N} \quad \text{i.e.} \quad d = \frac{L_n}{n_g}\left(1 - \frac{L_n}{L_t}\right).$$

For the various tracks, we find d values that fluctuate between 0.6-0.7 μ; of course this formula gives the same values for the length of a gap corresponding to a single undeveloped grain.

Histogram in Fig. 1 gives the distribution of groups according to their length for the track considered in Table I.

We note that the calculated value of d coincides with the maximum of

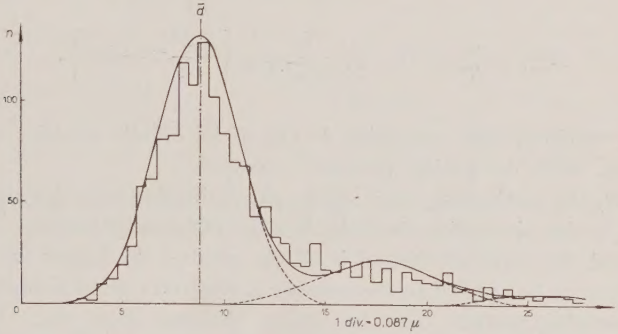

Fig. 1. – Distribution of groups according to their length for the track considered in Table I. Continuous line: $P(1)$ given by (6). d is the length of single grain-group calculated from (5).

the distribution curve. In fact, if we introduce in formula (4) the numerical value $p = 0.222$, we find that the number of single-grain groups is very much higher than that of the pluri-grain ones. Once d has been determined, the N value follows immediately.

For grain density measurements, the number of grains to be attributed to each group is the integer that is closest to the numerical value of the ratio group-length/d. Thus, single-grain groups are those with $0 < L \leqslant 3d/2$; two-grain groups, those with $3d/2 < L \leqslant 5d/2$; and so on.

In Table I (columns 1 and 2) the number of one, two, three, ... grain groups, deduced in this way, are compared to the corresponding numbers calculated from (4). The two sets of numbers agree fairly well.

From the histogram of Fig. 1, it is evident that the length of single-grain groups fluctuates about a mean value d: this indicates that the grains do not all have exactly the same diameter.

If we assume that all the grain groups with $L \leqslant d$ consist of one grain (an assumption wholly legitimate in the present case, and, in general, for all particles whose ionization is close to the minimum value), it becomes possible to test by means of our experimental data whether or not the length distribution is normal. This can be done by comparing the frequency of the reduced deviations with the Gauss function $\Theta(\varepsilon h)$.

The results of such a comparison are plotted in Fig 2; we have taken for the precision constant the value $h = 0.35$. As it is readily seen, the distribution can be considered Gaussian.

Groups due to a random association of grains, must therefore have lengths distributed about the values id, (precision constant $= h/\sqrt{i}$). The length distribution function of individual groups can then be expressed by

$$P(l) = \sum_i p^i (1-p)^2 \frac{h}{\sqrt{\pi i}} \exp\left[-\frac{h^2}{i}(l-id)^2\right], \tag{6}$$

i.e., as the weighted sum, according to (4), of an infinite number of Gaussian distributions, with decreasing precision constant.

However, the method outlined above, of attributing to each group a certain number of grains, is legitimate only if h is sufficiently large.

In general, the error introduced is all the greater, the higher the value of i. The correction to be applied to the number giving the i-grain groups, evaluated by the preceding method, can be readily deduced from (6). In Table I (column B) we give these corrected values. For tracks whose ionization is close to the minimum value, we find that the correction is small, and within experimental errors. For larger energy losses, however, it is not possible to deduce the h values directly from experimental data, obtained from measurements along the track, for, since n_2 is not much smaller the n_1, a certain number of two-grain groups are included among groups whose length is $\leqslant d$.

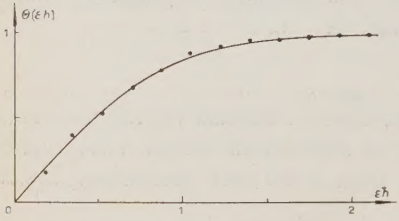

Fig. 2. – Frequency distribution of reduced deviations $\leqslant |\varepsilon h|$. Dotted line: Gaussian function $\Theta(\varepsilon h)$.

Following the same experimental technique, we have investigated a proton track of 18 000 μ in an Ilford G-5 (plate C) which had been exposed at 3 500 m a.s.l. The track ends in the emulsion; the energy corresponding to the initial part is roughly 75 MeV. The track was subdivided into sections, such that the increase of energy loss was 1 MeV/g cm^{-2} in each one. (The first 1 500 μ were discarded).

The histogram of Fig. 3 represents the length distribution of individual grain groups, and of the corresponding gaps, for the first section of the track. We note, that the gap distribution curve shows a marked displacement towards the smaller values; this indicates that there is a significant number of gaps whose length is less than the minimum length of the single-grain groups.

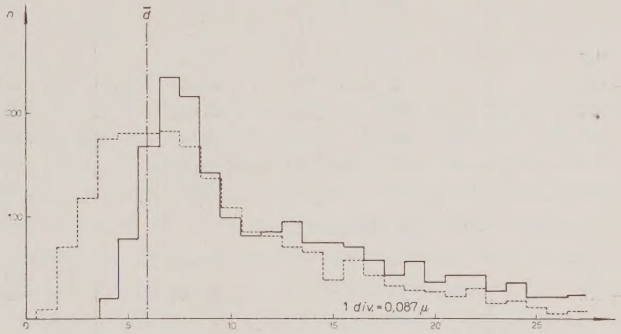

Fig. 3. – Proton of 75 MeV between 17 200 and 13 000 μ from the end. Solid histogram: length distribution of black groups; dotted histogram: length distribution of gaps.

The latter result is not consistent with the scheme we have been discussing, for if such a scheme were correct, the mean length, both of groups and gaps consisting of one grain, ought to be the same, and show the same distribution about this mean value.

In Table II, we have tabulated the d values (diameter values) deduced from (5), of both single-grain groups and gaps, for a number of tracks, chosen at random in plates A and B. Of these, some were more or less strictly collimated to the incident meson beam, and some not at all; we must therefore consider them as due to particles of different mass and energy.

The tracks were subdivided into sections such that the increase of energy loss along each did not exceed 1 MeV/g cm^{-2}. The various sections of all the tracks were then divided into groups, according to their p value, and the mean value of d, for each p interval, evaluated. These are precisely the d values given in Table II.

We find that there is a correlation between d and p; in order to render this relationship more evident, we have plotted d versus p in Fig. 4. As p increases, the mean diameter d decreases, while, according to our previous assumptions, it ought to remain constant. We also note that the numerical values of d, for tracks close to the minimum value of ionization, appear to be three or

TABLE II. – *Values of K and d for various intervals of p.*

p	n. of observed tracks	K	d (μ)
		PLATE *A*	
0.2 – 0.3	12	3,6	0,656
0.3 – 0.4	—	—	—
0.4 – 0.5	4	3.93	0.637
0.5 – 0.6	2	3.95	0.607
0.6 – 0.7	6	4.14	0.572
0.7 – 0.8	10	4.05	0.513
0.8 – 0.9	20	4.71	0.509
0.9 – 1.0	10	4.47	0.429
		PLATE *B*	
0,2 – 0.3	2	3.73	0.669
0.3 – 0.4	12	3.94	0.682
0.4 – 0.5	—	—	—
0.5 – 0.6	1	3.69	0.563
0.6 – 0.7	7	3.95	0.544
0.7 – 0.8	3	4.08	0.530
0.8 – 0.9	5	4.74	0,519
0.9 – 1.0	—	—	—

four times greater than the mean value of an unimpressioned grain measured under an electronic microscope in Ilford G.5 [5].

We must therefore conclude, either that as a result of the plate processing the grain sizes increase, or that the grains are not developed as a single entity, but rather in the form of unresolved clusters. The very simple considerations that follow, would appear to confirm this second hypothesis, without, however, excluding the first.

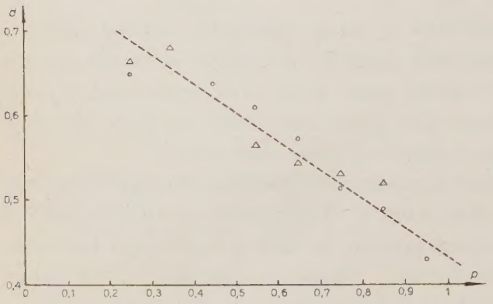

Fig. 4. – Values of d, calculated from (5), versus p. ○ Plate *A*. △ Plate *B*. Points represent the mean values of d, for 0.1 intervals of p, deduced from a number of tracks.

Let S be the shrinkage factor of the emulsion, v the average volume of a AgBr grain (radius 0.17 μ [5]) and v_0 the average volume of emulsion

[5] G. BARONI and C. CASTAGNOLI: *Nuovo Cimento*, **7**, 394 (1950).

corresponding to a single grain. We have

$$\frac{v_0}{v_0 - v} = S ,$$

and, assuming v_0 spherical, with diameter x_0

$$1 - \frac{1}{S} = \left(\frac{0.17}{x_0} \right)^3 .$$

If we assume $S = 2.7$, we find $x_0 = 0.2$ μ, that is to say, the layer of gelatine which separates the single grains, has a mean value of 0.03 μ. Now, even if we assume that the grain, once the plate has been processed, fills up the entire space which it has at its disposal, we still remain notably below the experimental value of d. This suggests that the black elements of the track which we consider to be single grains, are in reality unresolved clusters. We thus get indirect evidence of a developing process which not only affects the grains impressioned by the traversing particle, but also a certain number of other grains which happen to be neighbours to the directly affected ones.

The ratio $p = L_n/L_t$, therefore, gives the probability for a grain to be developed, either because it has been traversed by the ionizing particle, or in consequence of an induced process of development.

It is not possible, therefore, to establish an univocal relationship between grain density, as it is measured according to standard methods, and energy loss, if we do not first find a way of evaluating the percentage of black grains that owe their development to this induced process.

2. – New scheme for the process of track formation.

In the light of the above considerations, we now suggest a new scheme for the process of track formation.

a) The track, previous to development, is formed by a succession of similar, adjacent grains, of equal diameter, d_0, a certain number of which has been rendered directly developable by the passage of the particle (6).

b) For a given emulsion, the probability that a grain be rendered directly developable by the passage of a charged particle, is a function only of the energy loss of the particle.

c) The development results in the blackening of the impressioned grain, which is accompanied by an increase of its size, and by a process of induced development on neighbouring grains, even if these were previously unaffected.

d) Tentatively we specify point *c*) by assuming that two grains, rendered directly developable, will belong to two different black clusters only when they are separated by at least K non impressioned grains; we will further assume that K is a function of the development degree only.

This scheme certainly represents a better approximation to the truth, as it takes into account a process of induced developemnt, for which there is evidence from other sources; to do this we introduce a single new parameter K, but on the other hand we start from a directly measured value (5) $d_0 = 0.2\ \mu$ (6).

According to this scheme, the total number of grain groups, and hence of gaps, is given by

$$n_g = \pi(1-\pi)^K N\,, \tag{7}$$

where N is the total number of grains.

The number of gaps corresponding to i undeveloped grains is

$$n_i = \pi^2(1-\pi)^{K+i-1}N\,. \tag{8}$$

Therefore, the total number of undeveloped grains is

$$n' = \sum_i N\pi^2 i(1-\pi)^{K+i-1} = (1-\pi)^K N \tag{9}$$

and the total length of the gaps

$$L_b = Nd_0(1-\pi)^K\,. \tag{10}$$

We also have

$$L_t = Nd_0\,. \tag{11}$$

Since L_b, L_t and n are direct experimental data, from (7), (10), (11) it is possible to deduce the π and K values:

$$\begin{cases} \pi = \dfrac{n_g}{L_b}\, d_0 \\ K = \dfrac{\log L_b/L_t}{\log\,(1 - n_g d_0/L_b)} \end{cases} \tag{12}$$

(6) The assumption regarding the grain diameter appears legitimate, at least as an approximation, as electronic microscope measurements (5) have shown that in an undeveloped emulsion the grain diameters have a very narrow Gaussian distribution about a mean value of 0,17 μ (G5 Ilford). As our emulsions are Ilford G5, we have assumed $d_0 = 0.2\ \mu$.

If this scheme is correct, the value of π so deduced must be independent of the degree of development, under the assumption that the emulsion be not underdeveloped, i.e., that all impressioned grains be developed.

If this conditions is fulfilled, the parameter K, which characterizes the degree of development of a plate, must be constant for a uniformly developed plate.

It is probable that this way of considering the process of track formation is no longer legitimate, when the energy loss is very great, or when there is an appreciable probability for the particle to render develo pable grains not directly traversed, through knock-on secondaries, or other processes. In this case, one would expect to fin da correlation between K and π or p.

The displacement towards the smaller values of the gap length distribution in comparison to that of the black clusters (Fig. 3), is readily explained in the light of the above considerations.

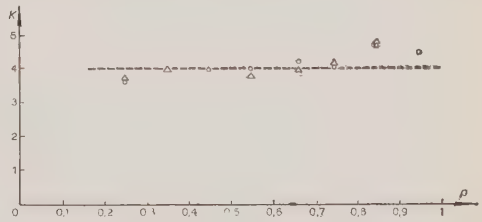

Fig. 5. – Values of K versus p. ○ Plate A. △ Plate B. Points represent the mean values of d, for 0.1 intervals of p deduced from a number of tracks.

It is evident, in fact, that the length of the smallest black group must, on the average, be K times greater than the shortest gap length.

In Table II (column 3) the K values are given for a certain number of tracks. It is evident from an analysis of Fig. 4 that, while for different particles p varies between 0.2 and 0.8, K fluctuates about a constant value. We have noted that the dispersion of K is a characteristic of the various plates, and of the homogeneity of their development.

It now becomes possible to account for the correlation between the d and p values, mentioned in section 1.

Formula (5) can be written

$$d = \frac{L_b}{n_a} \cdot p,$$

or ((10) and (7))

$$d = \frac{p}{\pi} d_0 .$$

d/d_0 represents, therefore, the ratio between the number of developed grains and the number of grains directly impressioned by the traversing particle.

According to our scheme, since $p = 1 - (1 - \pi)^K$,

$$\frac{d}{d_0} = \frac{p}{1 - (1 - p)^{1/K}} \tag{12}$$

and $\lim_{p \to 0} d_0/d = K$. The latter expression is fully consistent with the physical significance of K, for, if the p values are very small, the impressioned grains are widely separated, and consequently each of them corresponds to K developed grains.

3. – Normality of the grain distribution.

HODGSON (7) has noted that the distribution of the number of grains along a given track, presents a dispersion inferior to that which might be expected if the grains were distributed according to a Poisson law. In fact, in the case of a Poisson distribution the mean value of the number of grains, N, per L_μ of track, ought to be roughly equal to the square of the standard deviation σ of the series of N values, obtained by subdividing the entire track into sections of a given length, L. For a track whose grain density is close to the minimum value, HODGSON finds instead $\sigma^2 \sim 0.6\overline{N}$.

We have thought that it might be interesting to test whether our tracks appeared to behave in the same way; however, instead of studying the distribution of the number of grains, we have preferred to consider that of the grain groups, because the determination of this quantity, as we have seen, is more objective.

Therefore, we have further extended our analysis of some of the collimated tracks in plate *A*. Each of these was subdivided into sections of 100 μ, and the number of groups pertaining to each section was counted.

As a preliminary step, we have checked whether the succession of values for the different sections was « typical », i.e., whether its deviations were in accordance with a Gaussian law.

To this end, we have applied the following tests:

a) Computation of the precision constant of our distribution in three different ways, namely, by evaluating the mean, the standard, and the median deviations; if the distribution is a normal one, the results ought to be the same in the three cases.

(7) P. E. HODGSON: *Br. Journ. of App. Phys.*, **3**, 11 (1952).

b) Computation of the value of the ratio $K_3/K_2^{3/2}$ between the distribution's third moment and the 2/3 power of its second moment: for a normal distribution, this value ought to be comprised in a certain limiting interval.

c) Direct comparison between the frequency of the reduced deviations $\leqslant |\varepsilon h|$ and the Gaussian function, $\Theta(\varepsilon h)$ (see Fig. 6.)

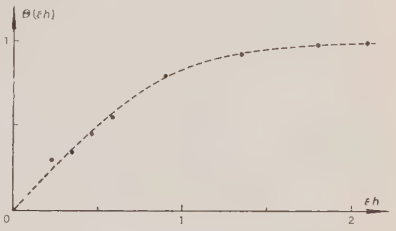

Fig. 6. – Frequency distribution of reduced deviations $\leqslant |\varepsilon h|$. Dotted line is the Gaussian function $\Theta(\varepsilon h)$.

In all our tracks, we have found a normal distribution of the number of groups counted along 100 μ sections. The numerical results of these tests, for one of the tracks, are given in Table II and Fig. 6. Finally, we have checked whether the distributions have a normal dispersion by comparing, for each track, the $\overline{n}_g$ values with the σ^2 evaluated from 24 sections of a 100 μ. Results are shown in Fig. 7: in the seven tests that were carried out, we have always found $\overline{n}_g > \sigma^2$.

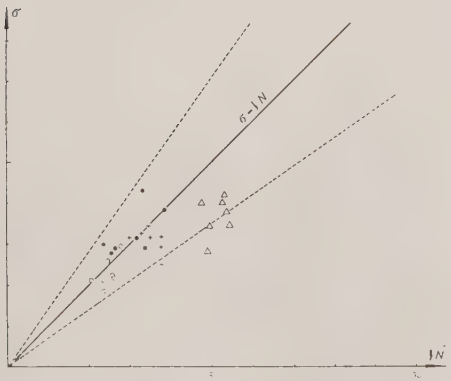

Fig. 7. – Dispersion of group distribution. Standard deviation σ versus $\sqrt{N}$. △ All groups ; ● Groups of length $\leqslant 0{,}7$ μ ; + Groups of length of $0.7\ \mu < l < 1.2\ \mu$; ○ Groups of length $\geqslant 1.2$ μ. Dotted lines give the limits for a 5% probability according the Snedecor's Table.

All the $\overline{n}_g/\sigma^2$ values are >1, and though a comparison with the corresponding values given in a Snedecor Table, with 5% probability, shows that the differences between n_g and σ^2 are significant only in three of our cases, we may infer that also the dispersion of our distribution is sub-normal.

This result appears perfectly consistent in the scheme we have outlined above. In this scheme, in fact, the random event is the activation of a grain produced by an ionizing particle, while the event « development of a grain » is correlated to the first by a certain law.

If each impressioned grain would give rise to one group, we should always find $N = \sigma^2$; but, as the induced development welds together in a single cluster two or more impressioned grains, when the number of grains not directly affected that separates them is less than K, the number of groups is smaller

than that of the impressioned grains; it is precisely this welding effect that reduces the fluctuations, and so renders the distribution sub-normal.

Assuming a Poisson distribution of the impressioned grains, the probability of finding two of these at a given distance l is given by

$$p(l)\,\mathrm{d}l = \exp\left[-l/\bar{l}\right]\mathrm{d}l\,,$$

where $\bar{l}$ is the mean distance between two impressioned grains. If we call λ the maximum distance between two impressioned grains, for which this welding in a single group may occur, the number of groups in a given length L is

$$n_g = n\left[1 - \frac{\int_0^{\lambda} p(l)\,\mathrm{d}l}{1 - e^{-n}}\right] = n\exp\left[-\frac{\lambda}{L}n\right],$$

where n is the number of impressioned grains contained in L. For one of our tracks, we have found $n = \pi N = 30$ grains/100μ, and $n_g = 23.5$ groups/100μ. This means that in the present instance

$$\exp\left[-\frac{\lambda}{100\ \mu}\cdot 30\right] = 0.785 \quad \text{i.e.} \quad \lambda \sim 0{,}8\ \mu\,.$$

As λ/d_0 is the constant K of our scheme, we again find the value $K = 4$, previously obtained by other means.

We may add, as a further confirmation of this way of considering the actual process of track formation, that if n_g' is the number of groups whose length is comprised within a rather narrow range, the distribution of the n_g' values must have a normal dispersion, because, by attributing to the group length a precise value, we eliminate the compensation between frequencies of small and large groups, to which the sub-normality of the n_g distribution is due: n_i' must therefore behave like a random variable. Fig. 7 shows the comparison between n_g' and the corresponding σ^2 for three different ranges of group lengths. We find that the dispersion of these distributions can be considered normal.

Following the older scheme outlined in section 1, it is rather difficult to account for the sub-normality of the grain-distribution observed by HODGSON. The interpretations that have been tentatively offered in Hodgson's paper do not appear to be fully satisfactory.

We think that such a sub-normality is a consequence of the fact that Hodgson counts as single impressioned grains the clots of length $\sim 0.8\ \mu$, while, according to our scheme, such clots should be considered groups of K developed grains (of which only one directly affected by the particle, and $K-1$ developed by an induction process). In fact, we have previously shown

TABLE III. – *Statistical analysis of the group distribution in a track of plate A.*

Total Length of track	μ 6500
Mean number of groups/100 μ	23.58
Variance σ^2	11.48
Mean Deviation $\varkappa$	2.88
Standard Deviation σ	3.63
Median Deviation ϱ	2.42
Precision constant $h = \frac{0.564}{\varkappa} =$	0.196
Precision constant $h = \frac{0.707}{\sigma} =$	0.195
Precision constant $h = \frac{0.477}{\varrho} =$	0.197

$\frac{k_3}{k_2^{3/2}} = -0.16$; Interval for a typical distribution: ± 0.3

that in a well developed Ilford G5 plate K is ~ 4 and $d_0 \sim 0.2\ \mu$, so that the length of a single group is $0.8\ \mu$.

In HODGSON's work, groups were counted as two or three grains according to their length (taking $d_0 \sim 0.8\ \mu$ as the diameter of a single grain); a tentative calculation, following our scheme and using a reasonable value of π, shows in fact that probably a high proportion of groups that were so counted as one grain actually contained two impressioned grains.

If this be the case, the directly measured grain density $\mu = 0.259$ grains/μ would turn out to be smaller than the true one. This is confirmed by his calculations on the distribution of the grain separations $\geqslant 1.33\ \mu$, i.e. with the practical exclusion of the two-grain groups (the groups of three or more grains are rare in a minimum track). Since in this way the mean diameter of the grain does not appear in the calculations, HODGSON gets a higher value for the grain density, namely $\mu' = 0.326$ grains/μ.

As pointed out by HODGSON, the problem is similar to that of estimating the efficiency of a counter of « dead time » λ. According to the relation of BLACKMAN and MICHIELS $\mu'/\mu = 1/(1 + \mu\lambda)$, whence HODGSON finds $\lambda = 0.8\ \mu$.

Indeed if most of the two-grain groups have been considered as a single grain, the « dead time » λ should be about $(K - 1)d_0$, i.e. the maximum distance between two grains observed as a single group; with $K \sim 4$, that would mean $\lambda \sim 0.6\ \mu$ in fair agreement with the direct result $\lambda = 0.8\ \mu$.

It is impossible to reach more than a semi-quantitative confirmation of our interpretation because HODGSON's data are not sufficient for a precise evaluation of π and K. The sub-normality of the grain distribution as pointed out by HODGSON, is a straightforward consequence of such a « dead time ».

4. – Final remarks.

Further investigations are now under way, in view of obtaining a more complete check of the validity of our scheme, and of determining the limitations of its applicability.

In particular, we propose to study the corrections to be applied to the experimental values of π and of K for tracks that are at an angle to the plane of the emulsion.

From a practical point of view, the new scheme presents the following advantages:

a) It is possible to establish an experimental relationship between energy loss and π, so that, once the latter quantity has been measured, it is possible to deduce the energy loss without calibrating the plate. This eliminates the uncertainty underlying the determination of minimum grain density, and also avoids the laborious scanning of a plate in view of finding long range tracks terminating in the emulsion, suitable for calibration.

b) The value of K and the limits within which it remains constant may be taken as an indication of the degree of development and also of its homogeneity.

c) For plates uniformly developed, K can be determined once for all, whereby π will follow simply from grain-group counting. This renders the collection of experimental data from a great number of tracks much quicker, objective, and statistically significant.

d) The determination of π and K may bring an appreciable contribution to the study of the latent image formation process, and at the same time give a clearer vision of the mechanism underlying the development process.

We wish to express our appreciation to Professor S. FRANCHETTI, Director of this Institute, for his valuable suggestions in connection to this work, and to Professor M. MANDÒ and Doctor T. FAZZINI for many helpful discussions.

Thanks are also due to Professor G. BERNARDINI, who sent us plates *A* and *B*.

RIASSUNTO

Viene esaminato il significato della densità dei granuli nelle tracce osservate in emulsioni nucleari. Da misure del numero e della lunghezza dei gruppi neri e delle lacune per tracce di diversa densità, si deduce che il diametro medio apparente dei gruppi di un solo granulo decresce al crescere della densità. Questo effetto viene attribuito all'influenza dello sviluppo. Confrontando le due distribuzioni delle lunghezze delle lacune e dei gruppi neri si deduce che il più piccolo diametro medio delle lacune è notevolmente inferiore di quello dei gruppi. Nell'ipotesi dell'esistenza di un processo di sviluppo indotto sui granuli adiacenti a quelli effettivamente impressionati dalla particella si propone un nuovo schema per la struttura della traccia. In questo nuovo schema si suppone che due granuli impressionati vengono osservati come appartenenti a due gruppi distinti solo se fra di essi sono compresi almeno K granuli non impressionati. Il parametro K caratterizza, per una data emulsione, il grado di sviluppo. Misurando il numero di gruppi, e la lunghezza totale delle lacune su una lunghezza data di traccia è possibile calcolare sia K che la probabilità μ di impressionamento di un granulo. Si constata sperimentalmente che K si mantiene costante in un ampio intervallo di densità. In questo nuovo schema sono facilmente spiegabili le divergenze sopra dette nonchè la sottonormalità della distribuzione del numero di granuli osservata da Hodgson. In una seconda parte di prossima pubblicazione verrà discussa la relazione fra π e la perdita di energia della particella.

A Covariant Formulation of the Non-Adiabatic Method for the Relativistic Two-Body Problem. (I).

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Summary. — The non-adiabatic treatment of the relativistic two-body problem can be done with the Tamm-Dancoff method, or with the Bethe-Salpeter equation. The first one is formally non covariant, and no radiative effects can be computed; the latter is covariant, but the physical interpretation of the « wave function » involved is not clear. In the present paper a covariant formulation of the Tamm-Dancoff method is found, with the purpose of performing the renormalization of the divergent terms, thus obtaining a consistent and unambiguous field-theoretical treatment of the two body bound state. In the present part I an integral equation up to fourth order terms in powers of the coupling constant is given for the representative of the state vector in the interaction representation describing two nucleons at different space-time points. In part II to be published shortly the renormalization procedure of such equation will be discussed.

1. – Introduction.

The study of the bound system of two particles interacting through a quantized field has been done so far in two different ways. With the so called adiabatic method an effective potential between stationary particles is calculated, and subsequently their motion is studied introducing the derived potential in a two body wave equation. This procedure is obviously non relativistic and in the case of intrinsically relativistic interactions, such as given by pseudoscalar meson theory does not even allow for the existence of a bound system [1]. For this reason a non adiabatic method has been proposed by

[1] M. M. Lévy: *Phys. Rev.*, **84**, 441 (1951).

TAMM ([2]) and subsequently by DANCOFF ([3]) (abbreviated in the following as T.D.) in which an approximate Schrödinger equation for the state vector to the second order in the coupling constant is derived from the neutral scalar meson theory without the introduction of the concept of potential energy. Along these lines BARONCINI has extended the method to the pseudoscalar theory ([4]), and LÉVY has taken in account higher order terms ([5]).

This method makes use of field theory in a non-covariant form and it appears impossible in this framework to use the modern techniques of FEYNMAN ([6]), and DYSON ([7]) to separate effects of self-mass and self-charge, and take properly in account contributions of higher order in powers of the coupling constant.

An alternative procedure has been proposed by BETHE and SALPETER ([8]) by means of which a covariant equation is derived for a 16-component « wave function » such that the use of renormalization technique becomes possible. The Bethe-Salpeter equation (abbreviated as B.S. in the following) has been shown by LOW and GELL-MANN ([9]) to follow from field theory, but the physical meaning of the « wave function » is not yet properly understood, and it is not known how to calculate mean values of physical observables others than the binding energy of the system. A detailed study of the neutron proton system has been done by LÉVY ([10]) using the T.D. method for the convergent processes, and the B.S. equation when renormalization of mass and charge has to be carried out. The B.S. « wave function » with the times of the two particles equalled has been compared with the probability amplitude of the state with only two nucleons in the T.D. treatment ([7]), and the two turn out to be equivalent; nevertheless, considering that the two descriptions are generally quite different, the legitimacy of choosing some convergent parts from the first one and other renormalized contributions from the second has not been really proved. It seems worthwhile to investigate whether a covariant form can be found for the T.D. method, such that both the contributions from convergent processes and those arising from radiative corrections can be consistently derived from the same theory ([11]).

([2]) I. TAMM: *Journ. Phys.* (*U.S.S.R*), **9**, 449 (1945).
([3]) S. M. DANCOFF, *Phys. Rev.*, **78**, 382 (1950).
([4]) D. BARONCINI: *Nuovo Cimento*, **9**, 642 (1952).
([5]) M. M. LÉVY: *Phys. Rev.*, **88**, 72 (1952).
([6]) R. P. FEYNMAN: *Phys. Rev.*, **76**, 749, 769 (1949) (referred to as FI, FII).
([7]) F. J. DYSON: *Phys. Rev.*, **75**, 486, 1736 (1949) (referred to as DI, DII).
([8]) E. SALPETER and H. A. BETHE: *Phys. Rev.*, **84**, 1232 (1951).
([9]) M. GELL-MANN and F. LÖW: *Phys. Rev.*, **84**, 350 (1951).
([10]) M. M. LÉVY: *Phys. Rev.*, **88**, 725 (1952).
([11]) According to a footnote in ref. ([5]) a similar attempt seems to have been made made by WIGHTMAN and SCHWEBER, but so far I have not been able to see their work.

In the present paper (part I) such a covariant form is worked out, while the application of a renormalization procedure will be discussed in a forthcoming part II.

2. – The lowest order T.D. approximation.

Let us consider the equation for the wave functional $\Psi(\sigma)$ in the interaction representation

$$\Psi(\sigma) = \Psi(-\infty) - i\int_{-\infty}^{\sigma} dx_1\, \mathcal{H}(x_1)\Psi(\sigma_1)\,. \tag{1}$$

We will treat the case of two nucleons interacting through a neutral pseudoscalar meson field with pseudoscalar coupling. The extension to the other (renormalizable) meson theories is quite straightforward. Thus we assume

$$\mathcal{H}(x) = iG\sum_{\lambda=N,P}\bar{\psi}_\lambda(x)\gamma_5^\lambda\psi_\lambda(x)\varphi(x) + \text{Renormalization terms}\,. \tag{2}$$

The symbols are well known and the suffix λ distinguishes the proton and neutron fields; we are not concerned at the moment with the renormalization counter terms ([12]).

The field operators, which in the interaction representation obey the homogeneous equations of motion, can be decomposed covariantly in positive and negative frequency parts ([13])

$$\left|\begin{aligned}
&\psi_\lambda^{(+)}(x) = \int\sum_r u_\lambda^r(p)a_\lambda^r(p)\delta(p^2+m^2)\eta^{(+)}(p)\,e^{ipx}\,\mathrm{d}^4p\,,\\
&\psi_\lambda^{(-)}(x) = \int\sum_r v_\lambda^r(p)b_\lambda^{r*}(p)\delta(p^2+m^2)\eta^{(-)}(p)\,e^{ipx}\,\mathrm{d}^4p\,,\\
&\varphi^{(+)}(x) = \int q(k)\delta(k^2+\mu^2)\eta^{(+)}(k)\,e^{ikx}\,\mathrm{d}^4k\,,\\
&\varphi^{(-)}(x) = \int q^*(k)\delta(k^2+\mu^2)\eta^{(-)}(k)\,e^{ikx}\,\mathrm{d}^4k\,,
\end{aligned}\right. \tag{3}$$

where

$$\left.\begin{aligned}\eta^{(+)}(k)&=1\\ \eta^{(-)}(k)&=0\end{aligned}\right\}\ k_0>0\,;\qquad \left.\begin{aligned}\eta^{(+)}(k)&=0\\ \eta^{(-)}(k)&=1\end{aligned}\right\}\ k_0<0\,. \tag{4}$$

([12]) P. T. Matthews: *Phil. Mag.*, **41**, 185 (1950).

([13]) The following notation will be used: x, p are four vectors of component $(p_0, \boldsymbol{p})$ $(t, \boldsymbol{r})$. Four dimensional scalar products are indicated with $px = -p_0t + \boldsymbol{p}\cdot\boldsymbol{r}$. Furthermore $E_p = (p^2+m^2)^{1/2}$ and $\omega_p = (p^2+\mu^2)^{1/2}$, where m and μ respectively the nucleon and meson masses.

As is well known $a(p)$, $b(p)$, which need be defined only for $p_0^2 = \boldsymbol{p}^2 + m^2 = E_p^2$, and $q(k)$ (defined for $k_0^2 = \boldsymbol{k}^2 + \mu^2 = \omega_k^2$) are destruction operators for nucleons, antinucleons and mesons respectively. It is also [14]

$$(5)\quad \begin{cases} [\psi_\lambda^{(+)}(x), \overline{\psi_\lambda^{(+)}}(x')]_+ = [\psi_\lambda^{(+)}(x), \bar{\psi}_\lambda^{(-)}(x')]_+ = -i\, S_\lambda^{(+)}(x-x'), \\ [\psi_\lambda^{(-)}(x), \overline{\psi_\lambda^{(-)}}(x')]_+ = [\psi_\lambda^{(-)}(x), \bar{\psi}_\lambda^{(+)}(x')]_+ = -i\, S_\lambda^{(-)}(x-x'), \\ [\varphi^{(+)}(x), \varphi^{(-)}(x')]_- = [\varphi^{(+)}(x), \varphi^{(+)*}(x')]_- = i\Delta^{(+)}(x-x') = -i\Delta^{(-)}(x'-x). \end{cases}$$

Our aim is to solve eq. (1) without using perturbation theory, which as it is well known, is not suitable to the treatment of the bound state problem.

In the formulation of the non adiabatic T.D. method one considers conventional eigenvectors of the free Hamiltonian (in the Schrödinger representation) in momentum space [15]

$$(6)\qquad A_\lambda^{r*}(\boldsymbol{p}) \ldots B_{\lambda'}^{s*}(\boldsymbol{p}') \ldots Q^*(\boldsymbol{k}) \rangle_0 ,$$

where $A_\lambda^{r*}(\boldsymbol{p})$, $B_{\lambda'}^{s}(\boldsymbol{p}')$, $Q^*(\boldsymbol{k})$ are creation operators of nucleons, antinucleons and mesons of momentum $\boldsymbol{p}$, $\boldsymbol{p}'$, $\boldsymbol{k}$ [16] and $\rangle_0$ is the vacuum state. We can write the interaction representation operators (3) in terms of these operators [17]:

$$(3')\quad \begin{cases} \psi_\lambda^{(+)}(x) = \int \sum_r u_\lambda^r(\boldsymbol{p}) A_\lambda^r(\boldsymbol{p}) \exp[i(\boldsymbol{pr} - E_p t)](m/E_p)^{1/2}\, \mathrm{d}^3\boldsymbol{p}, \\ \psi_\lambda^{(-)}(x) = \int \sum_r v_\lambda^r(\boldsymbol{p}) B_\lambda^{r*}(\boldsymbol{p}) \exp[-i(\boldsymbol{pr} - E_p t)](m/E_p)^{1/2}\, \mathrm{d}^3\boldsymbol{p}, \\ \varphi^{(+)}(x) = \int Q(\boldsymbol{k}) \exp[i(\boldsymbol{kr} - \omega_k t)](2\omega_k)^{-1/2}\, \mathrm{d}^3\boldsymbol{k}, \\ \varphi^{(-)}(x) = \int Q^*(\boldsymbol{k}) \exp[-i(\boldsymbol{kr} - \omega_k t)](2\omega_k)^{-1/2}\, \mathrm{d}^3\boldsymbol{k}. \end{cases}$$

[14] J. Schwinger: *Phys. Rev.*, **75**, 651 (1949).

[15] V. Fock: *Zeits. f. Phys.*, **75**, 622 (1932). See also P. A. M. Dirac: *The Principles of Quantum Mechanics* (Oxford, 1947), Chapter X.

[16] The operators A, B and Q are simply proportional to the a, b and q defined previously. Their commutation relations are well known:

$$[A^r(\boldsymbol{p}), A^{s*}(\boldsymbol{k})]_+ = \frac{1}{(2\pi)^3}\, \delta_{rs}\delta(\boldsymbol{p}-\boldsymbol{k})$$

$$[B^r(\boldsymbol{p}), B^{s*}(\boldsymbol{k})]_+ = \frac{1}{(2\pi)^3}\, \delta_{rs}\delta(\boldsymbol{p}-\boldsymbol{k})$$

$$[Q(\boldsymbol{p}), Q^*(\boldsymbol{k})]_- = \frac{1}{(2\pi)^3}\, \delta(\boldsymbol{p}-\boldsymbol{k}).$$

[17] W. Pauli: *Rev. Mod. Phys.*, **13**, 203 (1941), p. 224. The factor $\sqrt{m/E_p}$ derives from our normalization of the spinors: $\bar{u}u = 1$ instead or $u^*u = 1$.

Let us now define the quantity

$$(7) \qquad {}_0\langle \psi_P^{(+)}(x')\psi_N^{(+)}(x'')\,|\,\Psi(\sigma)\rangle = \mathcal{H}_\sigma(x', x'')\,.$$

From the point of view of relativistic covariance it is a 16-component spinor, and represents the probability amplitude that the state $\Psi(\sigma)$ describes a proton and a neutron at two different space-time points x', x'' respectively.

The lowest order T.D. approximation consists in assuming that only the representatives

$$(8) \qquad {}_0\langle A_P^r(\boldsymbol{p})A_N^s(\boldsymbol{p}')Q(\boldsymbol{k})\,|\,\Psi(\sigma)\rangle\,,$$

are different from zero besides the fundamental one

$${}_0\langle A_P^r(\boldsymbol{p})A_N^s(\boldsymbol{p}')\,|\,\Psi(\sigma)\rangle\,.$$

With this assumption, multiplying (1) to the left by ${}_0\langle \psi_P^{(+)}(x')\psi_N^{(+)}(x'')$ and making use of the commutation relations (5) one gets

$$(9) \qquad \mathcal{H}_\sigma(x', x'') =$$

$$= \mathcal{H}_{-\infty}(x', x'') - iG\int_{-\infty}^{\sigma} \mathrm{d}x_1 S_P^{(+)}(x' - x_1)\gamma_5^P\, {}_0\langle \psi_P^{(+)}(x_1)\psi_N^{(+)}(x'')\varphi^{(+)}(x_1)\,|\,\Psi(\sigma_1)\rangle -$$

$$- iG\int_{-\infty}^{\sigma} dx_1\, S_N^{(+)}(x'' - x_1)\gamma_5^N\, {}_0\langle \psi_P^{(+)}(x')\psi_N^{(+)}(x_1)\varphi^{(+)}(x_1)\,|\,\Psi(\sigma_1)\rangle\,.$$

As a first thing we will show that $\mathcal{H}_{-\infty}(x', x'')$ vanishes. This is essentially different from what happens in the collision problems (S matrix) and is peculiar of the bound state. In this treatment no adiabatic switching on of the interaction is postulated, and the state is considered to be bound even for $t \to -\infty$.

Remembering that $\Psi(\sigma)$ is an interaction representation state vector ([18]) we can write it, assuming σ to be a flat surface $t = \text{const.}$ in terms of a Schrödinger representation state vector Ψ_S

$$\Psi(t) = \exp\,[iH_0 t]\Psi_S\,,$$

where H_0 is the total free Hamiltonian. But Ψ_S describes a stationary state

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([18]) This argument is essentially the same of DII, paragraph II.

of total energy (in the center of mass system of the two particles) W, and its time dependence is given by

$$\Psi_S = \exp[-iWt]\Psi',$$

with Ψ' a time-independent state vector. Incidentally Ψ' is the state vector used by TAMM and DANCOFF.

Now $\mathcal{A}_t(x', x'')$ is a superposition of representatives

$$(10) \qquad {}_0\langle A_P^r(\boldsymbol{p}) A_N^s(\boldsymbol{p}') | \Psi(t)\rangle = {}_0\langle A_P^r(\boldsymbol{p}) A_N^s(\boldsymbol{p}') | \Psi'\rangle \exp[i(E_p + E_{p'} - W)t],$$

because ${}_0\langle A_P^r(\boldsymbol{p}) A_N^s(\boldsymbol{p}')$ is an eigenvector of H_0 with eigenvalue $E_p + E_{p'}$. The total energy of the bound system W will always satisfy

$$W < E_p + E_{p'},$$

being E_p, $E_{p'}$ free particle energies. Thus in the limit $t \to -\infty$ $\mathcal{A}_t(x', x'')$ vanishes if the usual prescription [19] is followed that $E_p + E_{p'} - W$ is considered to have an infinitesimal negative imaginary part. We will adhere to this prescription in all the calculations in this work.

One can obtain integral equations similar to (9) for the quantities ${}_0\langle \psi_P^{(+)}(x_1)\psi_N^{(+)}(x'')\varphi^{(+)}(x_1) | \Psi(\sigma_1)\rangle$ and ${}_0\langle \psi_P^{(+)}(x')\psi_N^{(+)}(x_1)\varphi^{(+)}(x_1) | \Psi(\sigma_1)\rangle$.

In the T.D. lowest order approximation it turns out that they are expressed only in terms of quantities of the type $\langle \psi_P^{(+)}\psi_N^{(+)} | \Psi\rangle$.

Substituting back in eq. (9) the following integral equation in the unknown function $\mathcal{A}_\sigma(x', x'')$ is derived

$$(11) \qquad \mathcal{A}_\sigma(x', x'') =$$

$$= -iG^2 \int_{-\infty}^{\sigma} \mathrm{d}x_1 \int_{-\infty}^{\sigma_1} \mathrm{d}x_2 \left\{ S_P^{(+)}(x' - x_2)\gamma_5^P S_N^{(+)}(x'' - x_1)\gamma_5^N \Delta^{(+)}(x_1 - x_2)\, \mathcal{A}_{\sigma_2}(x_2, x_1) + \right.$$

$$\left. + S_P^{(+)}(x' - x_1)\gamma_5^P S_N^{(+)}(x'' - x_2)\gamma_5^N \Delta^{(+)}(x_1, x_2)\, \mathcal{A}_{\sigma_2}(x_1 - x_2) \right\}.$$

Actually we have omitted from the r.h.s. the following terms

$$(12) \qquad -iG^2 \int_{-\infty}^{\sigma} \mathrm{d}x_1 \int_{-\infty}^{\sigma_1} \mathrm{d}x_2 \left\{ S_P^{(+)}(x' - x_1)\gamma_5^P S_P^{(+)}(x_1 - x_2)\gamma_5^P \Delta^{(+)}(x_1 - x_2)\, \mathcal{A}_{\sigma_2}(x_2, x'') + \right.$$

$$\left. + S_N^{(+)}(x'' - x_1)\gamma_5^N S_N^{(+)}(x_1 - x_2)\gamma_5^N \Delta^{(+)}(x_1 - x_2)\, \mathcal{A}_{\sigma_2}(x', x_2) \right\},$$

[19] See FI, paragraph 6.

They clearly describe self-energy effects (in terms of S matrix nomeclature) and they will be discussed together with other divergent contributions in the next section. Remembering that

$$(13)\qquad \left.\begin{aligned} S^{(+)}(x''-x') &= -\frac{i}{2}S_F(x'-x'') \\ \Delta^{(+)}(x''-x') &= -\frac{i}{2}\Delta_F(x'-x'') \end{aligned}\right\} t'<t'' ,$$

$$\left.\begin{aligned} S^{(-)}(x''-x') &= \frac{i}{2}S_F(x'-x'') \\ \Delta^{(-)}(x''-x') &= \frac{i}{2}\Delta_F(x'-x'') \end{aligned}\right\} t'>t'' ,$$

one can write

$$(14)\qquad \mathcal{R}_\sigma(x',x'') =$$

$$= +\frac{G^2}{8}\left\{\int_{-\infty}^{\sigma}\mathrm{d}x_1\int_{-\infty}^{\sigma_1}\mathrm{d}x_2 S_{FP}(x_1-x')\gamma_5^P S_{FN}(x_2-x'')\gamma_5^N\Delta_F(x_1-x_2)\,\mathcal{R}_{\sigma_2}(x_1,x_2) + \right.$$

$$\left. +\int_{-\infty}^{\sigma}\mathrm{d}x_1\int_{\sigma_1}^{\sigma}\mathrm{d}x_2 S_{FP}(x_1-x')\gamma_5^P S_{FN}(x_2-x'')\gamma_5^N\Delta_F(x_1-x_2)\,\mathcal{R}_{\sigma_1}(x_1,x_2)\right\} .$$

One notices that in the integrand contributions arise only from $\mathcal{R}_{\sigma_2}(x_1,x_2)$ with x_1 later than σ_2 and from $\mathcal{R}_{\sigma_1}(x_1,x_2)$ with x_2 later than σ_1; it should be pointed out that (14) is valid only for x', x'' later than σ, because only in this case it is identical to (11).

Let us now Fourier analyze equation (11) using expressions (3), and multiply both sides on the left with $\bar{u}_p^\nu(\boldsymbol{k})\bar{u}_N^{\nu'}(\boldsymbol{j})$. Remembering that

$$(15)\qquad \left\{\begin{aligned} &\bar{u}_\lambda^r(\boldsymbol{k})u_\lambda^s(\boldsymbol{k}) = \delta_{rs} \\ &\sum_r u_\lambda^r(\boldsymbol{k})\bar{u}_\lambda^r(\boldsymbol{k}) = \frac{-i\gamma^\lambda k + m}{2m} \\ &S^{(+)}(x'-x'') = \frac{-i}{(2\pi)^3}\int (i\gamma k - m)\delta(k^2+m^2)\eta^{(+)}(k)\exp\left[ik(x'-x'')\right]\mathrm{d}^4k \\ &\Delta^{(+)}(x'-x'') = \frac{-i}{(2\pi)^3}\int \delta(k^2+\mu^2)\eta^{(+)}(k)\exp\left[ik(x'-x'')\right]\mathrm{d}^4k , \end{aligned}\right.$$

one obtains

$$(16)\quad \left(\frac{E_k E_j}{4m^2}\right)^{1/2} {}_0\langle A_P^v(\boldsymbol{k}) A_N^w(\boldsymbol{j})\,|\,\Psi(\sigma)\rangle = \frac{G^2}{(2\pi)^9}\int_{-\infty}^{\sigma} \mathrm{d}x_1 \int_{-\infty}^{\sigma_1} \mathrm{d}x_2 \iiint \mathrm{d}^4h\,\mathrm{d}^4l\,\mathrm{d}^4p \times$$

$$\times \Big\{ \sum_{rs} \bar{u}_P^v(\boldsymbol{k})\gamma_5^P u_P^r(\boldsymbol{h})\bar{u}_N^w(\boldsymbol{j})\gamma_5^N u_N^s(\boldsymbol{l})\,\delta(h^2+m^2)\delta(l^2+m^2)\,\delta(p^2+\mu^2)\eta^{(+)}(h)\eta^{(+)}(l)\eta^{(+)}(p)\times$$

$$\times \exp\left[-ix_1(k-h-p)\right]\exp\left[-ix_2(j-l+p)\right](4E_k E_l m^2)^{1/2}\times$$

$$\times {}_0\langle A_P^r(\boldsymbol{h})A_N^s(\boldsymbol{l})\,|\,\Psi(\sigma_2)\rangle + \text{term } (k\longleftrightarrow j)\Big\}\,.$$

Performing the integrations (with σ_1 and σ flat surfaces) it is readily seen, making use of (10), that the time dependence disappears, leaving the following equation

$$(17)\quad {}_0\langle A_P^v(\boldsymbol{k})A_N^w(\boldsymbol{j})\,|\,\Psi'\rangle = -G^2\int \frac{\mathrm{d}^3\boldsymbol{p}}{2\omega_p}\Big\{\frac{\sum_{rs}\bar{U}_P^v(\boldsymbol{k})\gamma_5^P U_P^r(\boldsymbol{k}-\boldsymbol{p})\bar{U}_N^w(\boldsymbol{j})\gamma_5^N U_N^s(\boldsymbol{j}+\boldsymbol{p})}{(\omega_p+E_j+E_{k-p}-W)(E_j+E_k-W)}\times$$

$$\times {}_0\langle A_P^r(\boldsymbol{k}-\boldsymbol{p})A_N^s(\boldsymbol{j}+\boldsymbol{p})\,|\,\Psi'\rangle + \text{term } (\boldsymbol{j}\longleftrightarrow\boldsymbol{k})\Big\}\,,$$

with

$$U_\lambda^r(\boldsymbol{k}) = \left(\frac{m}{E_k}\right)^{1/2} u_\lambda^r(\boldsymbol{k})\,,$$

Equation (17) is identical to the T.D. integral equation ([20]). The relation between our time-dependent function $\mathcal{A}_\sigma(x', x'')$ in configuration space, and the T.D. representative of the stationary state vector in momentum space, is therefore quite clear.

3. – The divergent contributions to the G^2 approximation.

To discuss the terms given in (12) it is necessary to take into account also the other terms of order G^2, arising from components of the state vector

$$(18)\qquad {}_0\langle A_P^r(\boldsymbol{p})A_N^s(\boldsymbol{p}')A_\lambda^t(\boldsymbol{p}'')B_\lambda^v(\boldsymbol{j})Q(\boldsymbol{k})\,|\,\Psi(\sigma)\rangle\,,$$

([20]) See ref. ([3]) eq. (9), or ref. ([5]) eq. (7). The factor $(m/E)^{1/2}$ is again due to the different spinor normalization.

describing the presence of one nucleon pair. Using the same procedure as before one obtains the following terms

$$(19)\qquad iG^2\int_{-\infty}^{\sigma}\mathrm{d}x_1\int_{-\infty}^{\sigma_1}\mathrm{d}x_2\,\{\,S_N^{(+)}(x''-x_2)\gamma_5^N S_N^{(-)}(x_2-x_1)\gamma_5^N\Delta^{(+)}(x_1-x_2)\,\mathscr{R}_{\sigma_2}(x',x_1)\,+$$

$$+\,S_P^{(+)}(x'-x_2)\gamma_5^P S_P^{(-)}(x_2-x_1)\gamma_5^P\Delta^{(+)}(x_1-x_2)\,\mathscr{R}_{\sigma_2}(x_1,x'')\,\}\,,$$

$$(20)\qquad -iG^2\int_{-\infty}^{\sigma}\mathrm{d}x_1\int_{-\infty}^{\sigma_1}\mathrm{d}x_2\,\mathrm{Tr}\,\{\,S_P^{(+)}(x_1-x_2)\gamma_5^P S_P^{(-)}(x_2-x_1)\gamma_5^P\,+$$

$$+\,S_N^{(+)}(x_1-x_2)\gamma_5^N S_N^{(-)}(x_2-x_1)\gamma_5^N\,\}\cdot\Delta^{(+)}(x_1-x_2)\,\mathscr{R}_{\sigma_2}(x',x'')\,.$$

In the conventional formulation of the T.D. method, contributions from terms (12) (19) and (20) have always been dropped because there was no way of eliminating covariantly the divergences. An attempt to overcome this difficulty will be developed in part II with the use of a consistent renormalization procedure. It will be shown that finite parts can be separated from terms (12) and (19) that must be kept and added to the r.h.s. of equation (11). The treatment of term (20), which represents, in the language of the S matrix, a contribution from closed loops in the vacuum will be also discussed.

4. – The integral equation to order G^4.

Lévy has extended the T.D. method to derive higher order contributions, including other representatives of Ψ' in addition to (8) and (18). We will consider the following ones:

$$(21)\qquad \begin{cases} {}_0\langle A_P A_N QQ|\,\Psi\rangle \\ {}_0\langle A_P A_N A_\lambda B_\lambda|\,\Psi\rangle \\ {}_c\langle A_P A_N A_\lambda B_\lambda QQ|\,\Psi\rangle \\ {}_0\langle A_P A_N A_P B_P A_N B_N|\,\Psi\rangle \\ {}_0\langle A_P A_N A_P B_P A_N B_N QQ|\,\Psi\rangle\,. \end{cases}$$

The last one has not been included in Lévy's formulation, but has to be taken in account if the total contribution to order G^4 is required.

We will show that the complete integral equation, up to terms of order G^4 has the form [21]:

$$(22)\qquad \mathcal{A}_\sigma(x', x'') =$$

$$= \frac{G^4}{64}\Big\{ \int_{-\infty}^{\sigma} \mathrm{d}x_1 \int_{\sigma_1}^{\sigma} \mathrm{d}x_2 \int_{\sigma_1}^{\sigma} \mathrm{d}x_3 \int_{\sigma_1}^{\sigma} \mathrm{d}x_4 \left[K\mathcal{A}_{\sigma_1}(x_3, x_4) + G_P \mathcal{A}_{\sigma_1}(x_4, x'') + G_N \mathcal{A}_{\sigma_1}(x', x_4) \right] +$$

$$+ \int_{-\infty}^{\sigma} \mathrm{d}x_2 \int_{\sigma_2}^{\sigma} \mathrm{d}x_1 \int_{\sigma_2}^{\sigma} \mathrm{d}x_3 \int_{\sigma_2}^{\sigma} \mathrm{d}x_4 \left[K\mathcal{A}_{\sigma_2}(x_3, x_4) + G_P \mathcal{A}_{\sigma_2}(x_4, x'') + G_N \mathcal{A}_{\sigma_2}(x', x_4) \right] +$$

$$+ \int_{-\infty}^{\sigma} \mathrm{d}x_3 \int_{\sigma_3}^{\sigma} \mathrm{d}x_1 \int_{\sigma_3}^{\sigma} \mathrm{d}x_2 \int_{\sigma_3}^{\sigma} \mathrm{d}x_4 \left[K\mathcal{A}_{\sigma_3}(x_3, x_4) + G_P \mathcal{A}_{\sigma_3}(x_4, x'') + G_N \mathcal{A}_{\sigma_3}(x', x_4) \right] +$$

$$+ \int_{-\infty}^{\sigma} \mathrm{d}x_4 \int_{\sigma_4}^{\sigma} \mathrm{d}x_1 \int_{\sigma_4}^{\sigma} \mathrm{d}x_2 \int_{\sigma_4}^{\sigma} \mathrm{d}x_3 \left[K\mathcal{A}_{\sigma_4}(x_3, x_4) + G_P \mathcal{A}_{\sigma_4}(x_4, x'') + G_N \mathcal{A}_{\sigma_4}(x', x_4) \right] \Big\} .$$

The reason why no terms of order G^2 appear will be made clear immediately. We consider separately contributions with convergent kernels and those with divergent ones.

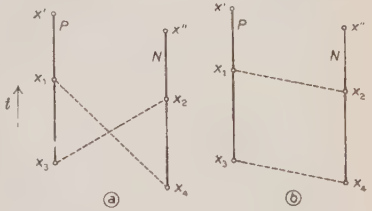

Fig. 1.

4·1. *Convergent kernels.* – The convergent contributions are all included in the kernel K (which contains *also* divergent terms). Carrying out some lengthy calculations with the method outlined in the previous sections one finds a very high number of terms [22] which can be classified using the Feynman graphical method (in configuration space). Summing together 24 of them one finds the following contribution to the kernel K:

$$(23)\qquad K_1^c = S_{FP}(x_1 - x')\gamma_5^P S_{FP}(x_3 - x_1)\gamma_5^P S_{FN}(x_2 - x'')\gamma_5^N S_{FN}(x_4 - x_2) \cdot$$
$$\cdot\, \gamma_5^N \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3) .$$

It clearly corresponds to the Feynman graph of fig. 1*a* [23]. One would expect a second group of terms to be collected to form the kernel of graph 1*b*, but it

---

[21] x', x'' are assumed to be later than σ.

[22] See Appendix.

[23] The 24 terms which contribute to the kernel (23) correspond to the 24 permutations in the order of x_1; x_2; x_3; x_4 in time (always remaining later than x', x'').

turns out that the four terms corresponding to [24]

$$x_1 > x_2 > x_3 > x_4$$
$$x_1 > x_2 > x_4 > x_3$$
$$x_2 > x_1 > x_3 > x_4$$
$$x_2 > x_1 > x_4 > x_3$$

are missing. This is easily understood considering that the missing terms can be obtained by iteration of (11) (or (14)). This equation is homogeneus (a peculiarity of the bound state problem), therefore if G_0^2 is one eigenvalue, $(G_0^2)^2$ will be the corresponding eigenvalue of the iterated kernel. It is convenient to transform in this way the G^2 terms in G^4 terms, adding them to the other ones, obtaining finally a second group of 24 terms which yield a contribution to K:

$$(24) \qquad K_2^c = S_{FP}(x_1 - x')\gamma_5^P S_{FP}(x_3 - x_1)\gamma_5^P S_{FN}(x_2 - x'') \cdot$$
$$\cdot \gamma_5^N S_{FN}(x_4 - x_2)\gamma_5^N \Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4) ,$$

described by the Feynman graph 1*b*. This is the reason why equation (22) does not contain G^2 terms. This situation is contrasted to the B.S. formalism, in which all contributions from reducible graphs (like 1*b*) are missing [25].

4·2. - *The divergent kernels.* – The divergent contributions are also of two types: those corresponding to irreducible graphs, which are genuine G^4 terms, and those obtained by iteration of the G^4 terms (12) and (19) added to the r.h.s. of equation (11). We will disregard graphs describing closed loops in the vacuum.

The divergent contributions to the kernel K from the first type of terms are:

$$(25) \qquad K_{1P}^d = S_{FP}(x_1 - x')\gamma_5^P S_{FP}(x_2 - x_1)\gamma_5^P S_{FP}(x_3 - x_2) \cdot$$
$$\cdot \gamma_5^P S_{FN}(x_4 - x'')\gamma_5^N \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) ,$$

and the corresponding

$$(26) \qquad K_{1N}^d = S_{FP}(x_3 - x')\gamma_5^P S_{FN}(x_1 - x'')\gamma_5^N S_{FN}(x_2 - x_1) \cdot$$
$$\cdot \gamma_5^N S_{FN}(x_4 - x_2)\gamma_5^N \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3) ,$$

[24] The sign $>$ means here later than.
[25] Reducible in the sense defined by Bethe and Salpeter, ref. [8].

obtained interchanging the proton and the neutron in the virtual process. The meson self-energy graph gives

$$(27)\qquad K^d_{2P} = -\,S_{FP}(x_3 - x')\gamma_5^P S_{FN}(x_4 - x'')\gamma_5^N \Delta_F(x_3 - x_1)\Delta_F(x_2 - x_4)\cdot \cdot \mathrm{Tr}\left[S_{FP}(x_2 - x_1)\gamma_5^P S_{FP}(x_1 - x_2)\gamma_5^P\right],$$

and a similar K^d_{2N} with a neutron loop instead of the proton loop.

The contributions to G_P are clearly proton self-energy terms

$$(28)\qquad G_{1P} = S_{FP}(x_1 - x')\gamma_5^P S_{FP}(x_2 - x_1)\gamma_5^P S_{FP}(x_3 - x_2)\cdot \cdot \gamma_5^P S_{FP}(x_4 - x_3)\gamma_5^P \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4),$$

$$(29)\qquad G_{2P} = S_{FP}(x_1 - x')\gamma_5^P S_{FP}(x_2 - x_1)\gamma_5^P S_{FP}(x_3 - x_2)\cdot \cdot \gamma_5^P S_{FP}(x_4 - x_3)\gamma_5^P \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3),$$

$$(30)\qquad G_{3P} = -\,S_{FP}(x_1 - x')\gamma_5^P S_{FP}(x_4 - x_1)\gamma_5^P \Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4)\cdot \cdot \mathrm{Tr}\left[S_{FP}(x_3 - x_2)\gamma_5^P S_{FP}(x_2 - x_3)\gamma_5^P + S_{FN}(x_3 - x_2)\gamma_5^N S_{FN}(x_2 - x_3)\gamma_5^N\right],$$

and similarly G_{1N}, G_{2N} and G_{3N} contribute to G_N.

From the iterated G^2 equation one has the contributions to K:

$$(31)\qquad K^d_{3P} = S_{FP}(x_1 - x')\gamma_5^P S_{FP}(x_2 - x_1)\gamma_5^P S_{FP}(x_3 - x_2)\cdot \cdot \gamma_5^P S_{FN}(x_4 - x'')\gamma_5^N \Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4),$$

$$(32)\qquad K^d_{4P} = S_{FP}(x_1 - x')\gamma_5^P S_{FP}(x_2 - x_1)\gamma_5^P S_{FP}(x_3 - x_2)\cdot \cdot \gamma_5^P S_{FN}(x_4 - x'')\gamma_5^N \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3),$$

and the correspondent terms K^d_{3N}, K^d_{4N}; while

$$(32)\qquad K^d_5 = S_{FP}(x_1 - x')\gamma_5^P S_{FP}(x_3 - x_1)\gamma_5^P S_{FN}(x_2 - x'')\cdot \cdot \gamma_5^N S_{FN}(x_4 - x_2)\gamma_5^N \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4),$$

is already symmetric in the neutron-proton exchange. Finally

$$(34)\qquad G_{3P} = S_{FP}(x_1 - x')\gamma_5^P S_{FP}(x_2 - x_1)\gamma_5^P S_{FP}(x_3 - x_2)\cdot \cdot \gamma_5^P S_{FP}(x_4 - x_3)\gamma_5^P \Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4),$$

contributes to G_P, and G_{3N} to G_N.

5. – Discussion.

The integral equation (22) is really meaningless because, except for K_1^c and K_2^c the other terms in K, G_P, G_N are all infinite: it will be shown in part II of this work how to separate covariantly some infinite parts of the kernels, absorbing them in unobservable mass and charge renormalizations. Finite contributions are left, which give, added to $K_c^c + K_2^c$ a completely divergence free integral equation.

The same thing can be done, of course, with the much simpler G^2 equation and could be done, at least in principle, for higher order equations in powers of G^2. It is not yet clear what to do with the divergent contributions from the disconnected vacuum closed loops: a detailed analysis of this point is in progress.

Although the present formulation of the relativistic two-body problem may seem involved and more complicated than the B.S. method, the advantage of having clearly defined the meaning of the quantities dealt with in the calculations must not be underrated. A fuller discussion will be published as a conclusion of part II.

The author wishes to thank Dr. N. Kemmer for stimulating discussions at the early stage of this work. In the same period a scholarship of the British Council is gratefully aknowledged.

Appendix

We will show here in detail how to derive one of the 24 terms contributing to the kernel (23) of equation (22).

Multiplying (1) to the left with [26] ${}_0\langle\psi_{P\alpha}^{(+)}(x_1)\psi_{N\beta}^{(+)}(x'')\varphi^{(+)}(x_1)$ one obtains:

$$(1A)\qquad {}_0\langle\psi_{P\alpha}^{(+)}(x_1)\psi_{N\beta}^{(+)}(x'')\varphi^{(+)}(x_1)\,|\,\Psi(\sigma_1)\rangle =$$

$$= iG\int_{\infty}^{\sigma_1} dx_2\, \Delta^{(+)}(x_1 - x_2)\,{}_0\langle\psi_{P\alpha}^{(+)}(x_1)\psi_{N\beta}^{(+)}(x'')\overline{\psi_{N\gamma}^{(-)}}(x_2)\,\psi_{N\delta}^{(+)}(x_2)\,|\,\Psi(\sigma_2)\rangle(\gamma_5^N)_{\gamma\delta} + \dots .$$

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[26] The greek indexes represent spinor components.

Similarly

$$(2A)\qquad {}_{\circ}\langle \psi^{(+)}_{P\alpha}(x_1)\psi^{(+)}_{N\beta}(x'')\overline{\psi^{(-)}_{N\gamma}}(x_2)\psi^{(+)}_{N\delta}(x_2)\,|\Psi(\sigma_2)\rangle =$$

$$= G\int_{-\infty}^{\sigma_2} \mathrm{d}x_3 S^{(+)}_{N\beta\varepsilon}(x''-x_3)(\gamma_5^N)_{\varepsilon\eta} S^{(-)}_{N\eta\gamma}(x_3-x_2)\,{}_{\circ}\langle \psi^{(+)}_{P\alpha}(x_1)\psi^{(+)}_{N\delta}(x_2)\varphi^{(+)}(x_3)\,|\Psi(\sigma_3)\rangle + \ldots$$

$$(3A)\qquad {}_{\circ}\langle \psi^{(+)}_{P\alpha}(x_1)\psi^{(+)}_{N\delta}(x_2)\varphi^{(+)}(x_3)\,|\Psi(\sigma_3)\rangle =$$

$$= G\int_{-\infty}^{\sigma_3} \mathrm{d}x_4\, S^{(+)}_{P\alpha\vartheta}(x_1-x_4)(\gamma_5^P)_{\vartheta\xi}\Delta^{(+)}(x_3-x_4)\,{}_{\circ}\langle \psi^{(+)}_{N\delta}(x_2)\psi^{(+)}_{P\xi}(x_4)\,|\Psi(\sigma_4)\rangle + \ldots .$$

Introducing (3A) in (2A); (2A) in (1A); and (1A) in the first term of the r.h.s. of (9) one obtains

$$(4A)\qquad \mathcal{A}_\sigma(x', x'') =$$

$$= G^4\int_{-\infty}^{\sigma}\mathrm{d}x_1\int_{-\infty}^{\sigma_1}\mathrm{d}x_2\int_{-\infty}^{\sigma_2}\mathrm{d}x_3\int_{-\infty}^{\sigma_3}\mathrm{d}x_4\, S_P^{(+)}(x'-x_1)\gamma_5^P S_P^{(+)}(x_1-x_4)\gamma_5^P S_N^{(+)}(x''-x_3)\gamma_5^N S_N^{(-)}(x_3-x_2)\gamma_5^N\cdot$$

$$\cdot\Delta^{(+)}(x_1-x_2)\Delta^{(+)}(x_3-x_4)\,\mathcal{A}_{\sigma_4}(x_4, x_2) + \ldots .$$

Using (13) and interchanging variables one has finally.

$$(5A)\qquad \mathcal{A}_\sigma(x'\,x'') = \frac{G^4}{64}\int_{-\infty}^{\sigma}\mathrm{d}x_1\int_{-\infty}^{\sigma_1}\mathrm{d}x_4\int_{-\infty}^{\sigma_4}\mathrm{d}x_2\int_{-\infty}^{\sigma_2}\mathrm{d}x_3\, S_{FP}(x_1-x')\gamma_5^P S_{FP}(x_3-x_1)\gamma_5^P\cdot$$

$$\cdot S_{FN}(x_2-x'')\gamma_5^N S_{FN}(x_4-x_2)\gamma_5^N \Delta_F(x_1-x_4)\Delta_F(x_2-x_3)\,\mathcal{A}_{\sigma_3}(x_3, x_4) \ldots .$$

This is the contribution of the kernel (23) to equation (22) for

$$x_1 > x_4 > x_2 > x_3 .$$

RIASSUNTO

La trattazione non adiabatica del problema relativistico dei due corpi può essere fatta con il metodo di Tamm-Dancoff, oppure per mezzo dell'equazione di Bethe e Salpeter. Il primo è in forma non covariante, e non si presta quindi al calcolo delle correzioni radiative; la seconda è covariante, ma non è chiara l'interpretazione fisica della « funzione d'onda » che vi figura. In questo lavoro il metodo di Tamm-Dancoff viene formulato in modo covariante, al fine di poter rinormalizzare i termini divergenti, e di ottenere quindi una trattazione del problema dei due corpi con l'uso della teoria dei campi, priva di ambiguità. In questa parte I si deduce un'equazione integrale contenente termini fino al quarto ordine nella costante di accoppiamento, nella quale la funzione incognita è la componente del vettore di stato in rappresentazione di interazione che descrive due nucleoni liberi in due punti diversi dello spazio-tempo. Nella parte II che verrà pubblicata prossimamente verrà discusso il procedimento di rinormalizzazione di tale equazione.

On the Angular Correlation of the Products of Nuclear Disintegrations

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(ricevuto il 12 Febbraio 1953)

Summary. — Angular and energy associations between charged particles emitted in nuclear disintegrations have been investigated, and a marked angular correlation for « small angles » (less than 30°) as well as a probable correlation in energy have been observed for particles having energies of the order of 100 MeV. In this same energy range, an angular correlation for « large angles » (120° to 180°) has also been observed. The possible explanations of the observed angular and energy associations are discussed.

The problem of the angular correlation between the products of nuclear disintegrations has been studied by a number of authors (1), in particular with reference to the correlation between the particles of low energy and between those of very high energy (« showers »). In this paper we will focus our attention on the angular distribution of the particles whose energy is in general below the energy of « shower » particles, but above that of the lower energy evaporation products.

A majority of the particles of « intermediate » energy are protons (2) and are believed to be ejected from the nucleus in the first part of the nucleonic cascade. It is probable, moreover, that a significant part of the nucleonic component in this energy range should be directly associated both with the

(1) M. Danysz, W. O. Lock and G. Yekutieli: *Nature*, **169**, 364 (1952); G. Lovera: *Nuovo Cimento*, **9**, 134, 857 (1952).; M. Della Corte and M. Giovannozzi: *Nuovo Cimento*, **8**, 741 (1951); P. E. Hogdson: *Phil Mag.*, **43**, 190 (1952); G. Belliboni and B. Vitale: *Nuovo Cimento*, **10**, 72 (1953).

(2) U. Camerini, P. H. Fowler, W. O. Lock and H. Muirhead: *Phil. Mag.*, **41**, 413 (1952).

evaporative phase of the excited nucleus ([3]) and with the reabsorption or the inelastic scattering of mesons produced in the primary interactions. In general it may be stated that, if the energy of nuclear excitation does not reach a very high value, the model of the nucleonic cascade gives results which are in satisfactory agreement with the energy spectrum of the particles in this energy range, as well as with their absolute intensity ([4]) ([5]).

For somewhat higher values of nuclear excitation energy it seems that this agreement no longer exists ([6]). This may be attributed to errors introduced into the theoretical results, by an incomplete knowledge of the differential collision cross-sections at high energies, by the difficulty of tracing the cascade inside the nucleus, or perhaps by the existence of some factor not considered in framing the model.

1. – For this investigation we have employed nuclear research emulsions (Ilford G5 400 μ thick) exposed to the cosmic radiation by a constant volume balloon flown at an altitude of 95 000 ft. for 4 hours and at a geomagnetic latitude of 55° N. The plates were kindly lent to us by Professor C. F. Powell, F.R.S.

We have found it convenient to classify the stars using the symbols N_b, N_g and N_s referring to the number of « black », « gray » and « shower » particle tracks respectively, with N_h being defined as the sum $N_b + N_g$. We have called the mean number of grains per 50 microns the « grain density » of the track considered.

From a total of 5 000 disintegrations, 220 were selected, satisfying the following conditions:

a) The existence of a track with a minimum grain density appearing in the upper hemisphere. This track was considered, if it seemed acceptable, as being due to the particle producing the nuclear disintegration.

b) $N_h \geqslant 5$, in order to assure that most of the stars were due to the disintegration of silver or bromine nuclei.

c) $N_g \geqslant 2$.

We have considered in this investigation as « grey » tracks those tracks having a grain density between 2 and 5 times the minimum value. This restricted interval of grain density was chosen in order to minimize the number of included « shower » particles, and also in order to be able to include those

([3]) F. Ferrari and C. Villi: *Nuovo Cimento*, **9**, 487 (1952).

([4]) G. Bernardini, E. T. Booth and S. J. Lindenbaum: *Phys. Rev.*, **83**, 669 (1951); **85**, 824 (1952); **88**, 1017 (1952).

([5]) G. Puppi, V. De Sabbata and M. Manaresi: *Nuovo Cimento*, **9**, 726 (1952).

([6]) G. Bernardini, G. Cortini and A. Manfredini: *Phys. Rev.*, **79**, 952 (1950).

« grey » tracks whose angle with respect to the plane of the emulsion was large.

Moreover we have not considered stars having their centers very near the glass or the top surface of the emulsion or being located in a badly distorted part of the plate.

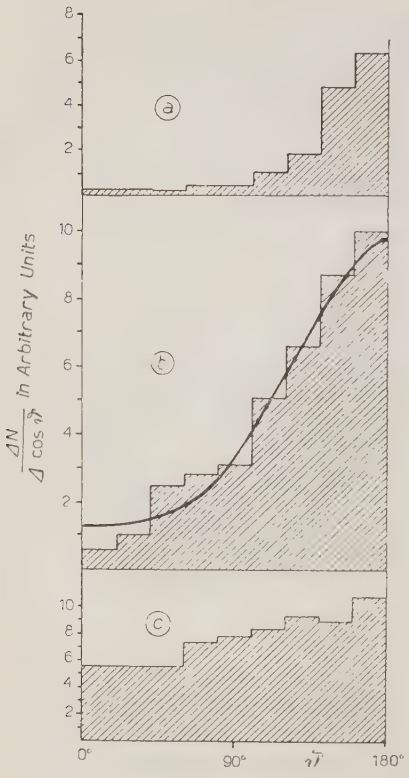

Fig. 1. – Frequency, per unit solid angle, of angles formed by the « primary » track and the tracks of « shower » particles, hist. *a*); « grey » particles, hist. *b*); and « black » particles, hist. *c*). The curve of « *b* » is an analytical approximation of hist. *b*).

For each track of the stars thus selected the zenithal and azimuthal angles as well as its grain density were measured. The values of the grain density have been corrected for their apparent increase due to the inclination of the tracks with respect to the plane of the emulsion.

Very steep tracks which were not completely « black » (several gaps over their length), were considered as being produced by particles with ionization loss similar to that of « grey » tracks, and were given a grain density of 105 grains per 50 microns. Using this value, a corrected value for the grain density was determined.

As a control in the use of the latter method we have compared the number of grey tracks forming angles less than 45 degrees with the plane of the emulsion, with that of tracks forming angles greater than 45 degrees. The grain density of the tracks in the first group may be measured, and moreover, because of the large angular dispersion of the primaries at 95 000 ft. it may be assumed that the angular distribution in space of the « grey » tracks is isotropic. The ratio of the number of tracks in each group divided by the corresponding solid angles should be equal, and this fact was verified. This indicates that no significant error in the classification of tracks in the various intervals of grain density has been made. In any case the number of tracks having large dip, and therefore a doubtful classification, was small.

For each interval of grain density we have determined the frequencies of the angles included between each track and the track of the particle pro-

ducing the disintegration. These frequency distributions are presented in Fig. 1, where histograms *a*, *b* and *c* are, respectively, the distributions for « showers », « grey » and « black » tracks. These values have been normalized to represent the number of tracks per unit solid angle.

For each star the angles between the different « grey » tracks have been determined. The number of these angles in intervals of 10 degrees are presented in the first line of Table I. Their frequency per unit solid angle is shown in fig. 2. The two values of the frequency for the intervals: 0° to 7° and 7° to 10°, listed in the third line of Table I, have been obtained from a larger number of stars in a way that will be discussed later.

TABLE I.

Angular interval	Observed	Calculated	Angular interval	Observed	Calculated
0° – 10°	23	12.7	0° – 7°	29	12.0
10° – 20°	43	36.6	7° – 10°	20	11.9
20° – 30°	64	57.7			
30° – 40°	67	76.4			
40° – 50°	89	90.9			
50° – 60°	93	101.8			
60° – 70°	81	108.4			
70° – 80°	106	111.2			
80° – 90°	125	111.9			
90° – 100°	94	107.5			
100° – 110°	85	93.5			
110° – 120°	88	90.2			
120° – 130°	77	67.5			
130° – 140°	77	63.7			
140° – 150°	46	49.2			
150° – 160°	44	34.4			
160° – 170°	31	19.9			
170° – 180°	7	6.5			

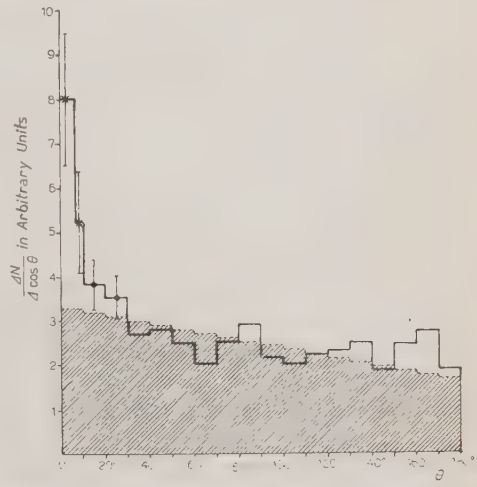

Fig. 2. – Frequency per unit solid angle of included angles between « grey » tracks; experimental —, and calculated assuming a random distribution ▨.

Were the angular distribution of tracks isotropic, the calculated distribution would appear as a horizontal line in fig. 2. This is not however to be expected, because the forward projection of the « grey » tracks, indicated in fig. 1*b*, tends to make the number of small angles larger. We have calculated

the expected frequencies of angles assuming that the tracks were independent from one another and that they had an angular distribution as shown in fig. 1*b*.

It may be shown that

$$P(\theta)\,\mathrm{d}(\theta) = \frac{1}{2\pi}\int_0^{\pi} \mathrm{d}\vartheta \sin\vartheta f(\vartheta) \int_0^{2\pi} \mathrm{d}\varphi' f(\cos\vartheta\cos\theta + \sin\vartheta\cos\varphi'\sin\theta)\sin\theta\,\mathrm{d}\theta$$

where $P(\theta)\,\mathrm{d}(\theta)$ is the probability of occurrance of an angle between two tracks having a value between θ and $\theta+\mathrm{d}\theta$ and where $f(\vartheta)$ is a function that approximates the distribution of angles formed by the « grey » tracks and the primary of the star. For the function $f(\vartheta)$ we have used the following form:

$$f(\vartheta) = A + B\cos\vartheta + C\cos^2\vartheta\,.$$

The evaluation of the constants A, B and C was made so that the curve of fig. 1*b* representing $f(\vartheta)$, would closely correspond to the histogram of the same figure.

The values of $P(\theta)$, as calculated on a basis of the total number of « grey » tracks, are listed in line 2 of Table I; and the same values divided by their corresponding solid angles are plotted as the « hatched » histogram of fig. 2.

From an examination of Table I and fig. 2, the frequency of the angles less than 30° appears to be favored with respect to the calculated distribution.

This result could possibly be attributed to the selection of stars of different types (thus tracks with a different angular distribution with respect to the primary would be included) or, perhaps, to the difference in the angular distribution of tracks of different energy. We have therefore treated the angular distribution similar to that of fig. 1*b* separately for stars of different types. We have considered the angular distribution of the « grey » tracks for stars with N_h greater than and less than 14 and for stars with N_s greater than or less than 2. We have also considered separately these distributions for tracks having a grain density between 2 and 3.5 and between 3.5 and 5 times the value of the minimum.

These distributions are rather similar and if the small differences between them and the distribution of fig. 1*b* are taken into account, the frequency of angles previously calculated is not significantly modified.

In order to confirm the preponderance of small angles, as previously indicated, we have examined 546 stars produced in the same plates as those first studied, each with $N_h \geqslant 10$ and having at least two « grey » tracks. We have considered for these stars the cases in which there were two « grey » tracks longer than 500 microns forming an angle of less than 10°.

Assuming that these stars have an anisotropic angular distribution for « grey » tracks analogous to that of fig. 1*b*, we have calculated also for these

stars the frequency of occurrence of the different angles. For the 766 stars considered, the observed frequency of angles in the intervals 0° to 7° and 7° to 10° are listed on the third line of Table I. The corresponding calculated values are shown on the fourth line of this same Table. These values have also been used to determine the corresponding points of the histogram of fig. 2.

A similar analysis was made for the angles between a « black » and a « grey » track and for those between two « black » tracks. With the number of angles available no significant difference between the experimental values and those calculated was found.

We have not, on the other hand, made a similar study for the tracks produced by « shower » particles. The angular distribution of these particles is strongly correlated with the energy of the primary particle and, in that the primaries of the stars have a wide energy spread, we maintain that the analysis in this case would give results very difficult to interpret.

2. – In this section we will discuss the result of a detailed study of the pairs of grey tracks forming angles less than 10°. The particles should have a high probability of being associated.

For these pairs, the grains of each track have been counted using a convention such that the grain density could be consistently determined. The average ionization of these tracks was not found to be appreciably different from that of the other grey tracks. For nine of these pairs the grain density of the two tracks was very similar (equal to a difference in energy of less than 12 MeV, assuming that the particles producing the tracks are protons). Assuming a random distribution of ionization values, only 4 pairs of particles with a difference of ionization of this order would have been expected.

A more concrete indication of such a correlation may be deduced from the histograms of fig. 3. In order to construct these histograms the tracks were divided into two groups, *A* and *B*. Group *A* includes those tracks having grain density between 2 and 3.5 times the minimum value, and group *B* those having grain density between 3.5 and 5 times the minimum. Histogram I of this figure represents, for each star considered separately, the frequency of angles included between tracks of the same group. Histogram II on the other hand, shows the frequency distribution of angles formed by a track of group *B* with a track of group *A*. Histogram III represents the calculated frequencies assuming a random distribution of included angles.

In comparing the two histograms I and II, the frequencies of angles between 0° and 20° are seen to be markedly different. The angles formed by tracks with similar grain density are seen to have a comparatively large frequency of occurrence in this angular interval, whereas for tracks of rather different grain density no significant difference is noted between the calculated

and the experimental values. We may therefore affirm, with reasonable certainty, the existence of an energy correlation between the particles emitted in « small-angle » pairs from nuclear disintegrations.

A detailed analysis of the « large-angle » effect, evident in histogram II of fig. 3, will be discussed in section 3.

Concerning the identification of the particles emitted in narrow-angle pairs, grain-density and scattering measurements have shown that they are not mesons, and in fact indicate that they have approximately protonic mass.

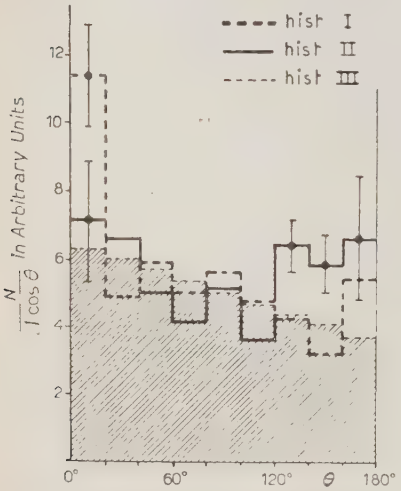

Fig. 3. – Frequency per unit solid angle of included angles between « grey » tracks of groups A (2 to 3.5× min.) and B (3.5 to 5 min.); angles between tracks of the same group, hist. I; angles between tracks of different groups, hist. II. Hist. III is the calculated distribution.

Comparing the experimentally determined frequencies of angles, as indicated in Table I, with the calculated values, it can be seen that the tracks emitted in pairs form $\sim 7\%$ of the total number of tracks. Such a value should perhaps be given as a lower limit because of the masking effect caused by non-correlated angles. This effect only becomes important, however, for comparatively large angles.

3. – In the previous sections evidence has been presented to confirm the existence of an angular correlation for intermediate energy particles. This effect is shown graphically in fig. 2. The same figure suggests that an angular correlation between particles forming angles of $\sim 180°$ also exists. This correlation for large angles may be expected from the consideration that slow π^+-mesons reabsorbed in the nucleus by a P—N pair can result in the emission at about 180° of two protons with energies of ~ 60 MeV (7). We have therefore selected tracks having a grain density between 3.5 and 5 times the minimum. For protons, these grain densities would correspond to 100 and 50 MeV, respectively. Using the tracks so selected and applying the methods used in constructing the histogram of fig. 2, the frequency distribution of the included angles between these tracks has been calculated, and is presented in fig. 4. In this figure the existence

(7) H. Byfield, J. Kessler and L. M. Lederman: *Phys. Rev.*, **86**, 17 (1952).

of a very large number of angles in the interval between 160° and 180° is observed, together with the comparatively large frequency of angles less than 20°.

Additional information on the process of π meson absorption may be gleaned from the angular correlation for large angles as observed in histogram II of fig. 3. In constructing this histogram the tracks used were divided into two grain density groups, A and B, and, for each event, the angles between a track of one group and one of the other were determined. Assuming that the particles producing these tracks were protons, their mean energy would be ~ 150 and ~ 70 MeV, respectively. The relatively large frequency of angles in the range from 120 to 180°, evident in the histogram II, can be interpreted as being due to the process of absorption by $P - N$ pairs of π^+-mesons with kinetic energies of the order of 100 MeV. Thus the two protons emerging from the nucleus would not be emitted in directions diametrically opposed, and would generally have different energies.

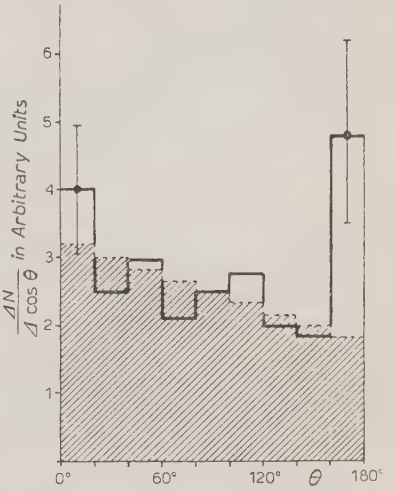

Fig. 4. – Frequency per unit solid angle of included angles between « grey » tracks of the grain density group B. The « hatched » histogram is the calculated distribution.

4. – Following the research previously described we have initiated an experiment in order to study specifically the existence of energy correlations beween particles of « intermediate » energy originating in a nuclear disintegration. For this investigation we have selected 103 nuclear disintegrations having at least 2 tracks with grain densities between 2.3 and 6.1 times the minimum value, and being longer than 500 microns.

We have endeavored to determine the grain density of these tracks using a method that would minimize any subjectivness in the measurement, thus assuring that the grain density of each track would be determined to an accuracy of at least 4%.

Assuming that these tracks were produced by protons, we have evaluated the energy differences between the particles emitted from the same disintegration. The frequencies of such differences have been compared with the corresponding frequencies as calculated. For the calculation we have assumed that the energy of the particles emitted from a single disintegration are randomly distributed, the frequencies of occurrence being determined by their energy spectra.

In fig. 5 the frequencies of the energy differences observed and the corresponding calculated values are shown. From these results and from analogous results obtained for the differences in ionization it may be stated that, if there exists an energy correlation between the particles emitted in the same disintegration, this correlation does not modify appreciably the random distribution in energy of the particles.

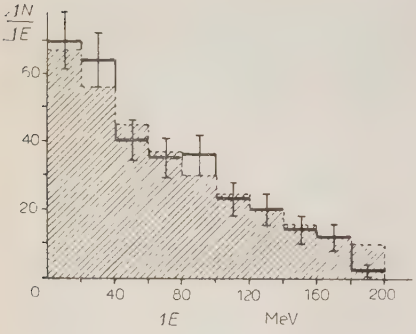

Fig. 5. - The number of «grey-track» pairs as a function of their difference in energy, calculated —; assuming a random distribution of energies ▨.

The possible of a correlation in energy between the particles whose angular divergence is small is not, however, contradicted by this last result; for resulting from the particular selection used in this latter analysis, the observed number of small-angle pairs is a very small percentage of the total number of tracks used.

Utilizing the values of the grain density obtained in this part of the experiment and applying the proton-deuteron-triton ratios at different energies as given by CAMERINI *et al.* [8] we have determined the energy spectrum of particles in the energy range from 40 to 250 MeV (fig. 6). A secondary maximum in the energy spectrum, observed also by G. BELLIBONI *et al.* [9], is seen to be centered about the value of 150 MeV. Its existence is held to be independent of any systematic error in the energy evaluation.

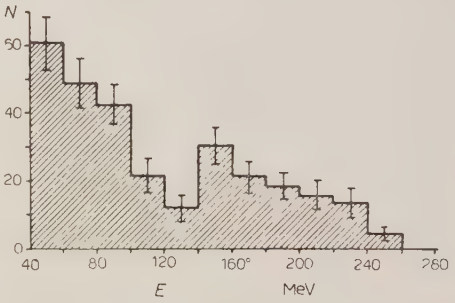

Fig. 6. – Differential energy spectrum for «grey» particles emitted in nuclear disintegrations.

Conclusion.

The results of this experiment seem to affirm the existence of an effect of angular association and a probable energy association for intermediate energy particles emitted from nuclear disintegrations in narrow angle pairs.

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[8] See reference [2].
[9] See reference [1].

A possible interpretation of this effect would be that the angularly correlated grey tracks are due to the splitting of alpha-particles or larger nuclear groups emitted in nuclear disintegrations. It seems reasonable to think that, if there exists the possibility of emission of alpha-particles and heavier nuclei in the ground state ([10]) ([11]), there also exists a certain probability that they are emitted in an excited state and, during the process of emission, disintegrate.

Using the results of SÖRENSEN ([10]) it is possible to calculate that the number of alpha-particles with energy greater than 70 MeV emitted from the type of stars that we have considered is $\sim 5\%$ of the total number of grey prongs. This value, when compared with the value we have obtained for the ratio of angularly associated tracks to the total number of grey tracks, would indicate that alpha-particles ejected in excited states have approximately the same frequency of emission as those ejected in the ground state.

The results presented in section **3** of this paper, indicate the existence of a correlation between particles having energies of the order of 70 MeV emitted in opposite directions from a nucleus. This may be considered as a direct proof of the frequent occurrence of a process in nuclei where a meson after its generation is absorbed by a $N - P$ pair with the resulting emission at 180 degrees of two protons each with an energy of ~ 60 MeV. If we consider the effect of « large angle » correlation as represented by figures 3 and 4 to be real and to be satisfactorily explained by π^--meson absorption, we may estimate the order of magnitude of the nuclear excitation energy attributable to this capture process. We have assumed for this calculation that in the process of π^--meson absorption by nuclei, only $\sim 10\ \%$ of the cases result in the direct emission of two « fast » protons at 180°. We have also tentatively assumed that the absorption of π^+, π^- and π^0-mesons contributes equally to the nuclear excitation. Using these assumptions together with the data presented in fig. 3 and 4, we have estimated the average energy of excitation due to this process to be ~ 800 MeV, corresponding to an absorption of ~ 3 charged mesons per star. This may be compared with an average energy of excitation of $\sim 1\,300$ MeV and an average number of emitted charged mesons equal to 1.5, for the stars considered in this analysis ([12]).

We would like to take this opportunity to thank Professor A. ROSTAGNI for his continued interest and encouragement during the course of this research, Professor N. DALLAPORTA and Dr. A. KIND for the assistance given us in

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([10]) S. O. C. SORENSEN: *Phil. Mag.*, **47**, 188 (1951); *Thesis*, Oslo (1951).

([11]) S. CRUSSARD: *Compt. Rend. Acad. Sci*, **231**, 141 (9150).

([12]) R. H. BROWN, U. CAMERINI, P. H. FOWLER, H. HEITLER, D. T. KING and C. F. POWELL: *Phil. Mag.*, **40**, 862 (1949).

calculating the statistical distributions and for their helpful suggestions and advise, and Dr. G. BELLIBONI and Dr. G. QUARENI for valuable criticism and discussion.

We would also like to thank Mr. A. BERNARDI for his intelligent collaboration in the analysis of events used in this investigation.

RIASSUNTO

È stata effettuata una ricerca mirante a mettere in luce l'esistenza di correlazione tra le particelle di media energia emergenti da una stessa disintegrazione nucleare. A tale scopo sono state utilizzate disintegrazioni prodotte da particelle di carica unitaria della radiazione cosmica in emulsioni nucleari portate ad un'altezza di 32 km s.l.m. È stato riscontrato in primo luogo un pronunciato effetto di associazione a piccoli angoli per tracce aventi densità di grani compresa tra 2 e 5 volte il valore minimo (energia delle particelle compresa tra 250 e 50 MeV se considerate protoni). Sembra inoltre possibile affermare che le tracce formanti coppie di piccolo angolo siano prodotte da particelle di energia simile. Tali effetti vengono interpretati con l'ipotesi che queste particelle formanti coppia siano emesse come risultato della frattura di particelle α veloci o di nuclei più pesanti emessi nel corso della disintegrazione. Un altro effetto messo in evidenza è la relativamente elevata frequenza di coppie di tracce formanti angoli di circa 180°. Interpretando tale effetto come dovuto al riassorbimento di mesoni π^+ da parte di coppie P-N, è possibile stimare il contributo fornito dal riassorbimento dei mesoni π all'energia di eccitazione del nucleo. Ne risulta che questo processo può essere considerato il fondamentale responsabile dell'eccitazione nucleare nei casi in cui tale eccitazione raggiunga valori elevati.

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Cloud Chamber Study of Cosmic Ray Electronic Showers under Dense Materials (I).

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(ricevuto il 17 Febbraio 1953)

Summary. — Pictures of electronic showers have been taken by means of a counter controlled cloud chamber. This 1st paper deals essentially with the methods employed in analysing the collected data. As an example, the results obtained with a lead plate 2 cm thick inside the chamber are described. They include angular distributions, distributions according to the number of electrons, ratio of photon initiated to electron initiated showers.

1. – Introduction.

It is well known that, although a great amount of work has been devoted —especially in the past—to the phenomenon of electronic showers, quantitative information about them is still scanty.

On the other hand, theoretical work on the subject has been pursued actively by several authors, even quite recently. Therefore it has been thought worth while to collect further data which could be used to check theoretical results. In this paper, however we shall limit ourselves to state essentially the methods employed in analysing the data, on the example of those obtained with a lead scatterer 2 cm thick.

The experiment was intended especially for the study of angular distributions of electrons but other data can be derived as well, such as the distribution of showers according to the number of electrons and the ratio of the photon initiated to the electron initiated ones. Moreover, the comparison between the various thicknesses and materials may also prove interesting.

2. – The experimental arrangement.

The arrangement employed consists of a cylindrical cloud chamber 25 cm in diameter with a plate L of the material under study inside it.

The expansion is controlled by fourfold coincidences between the counters $C_1 \ldots C_4$ (Maze type) located under the chamber (fig. 1). The probability of recording a shower is thus independent of the existence of any ionizing primary.

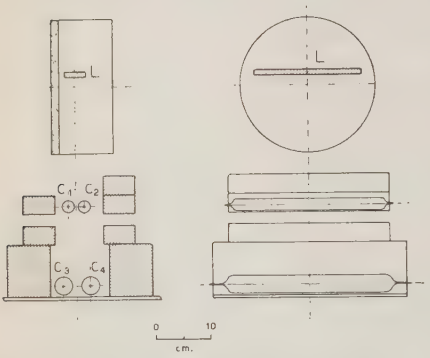

Fig. 1. – The experimental arrangement.

Pictures were taken along the axis of the chamber and at 15° from it by means of two Leica cameras. Illumination was made by a single Philips flash and a cylindrical lens.

About 6000 pairs of pictures were obtained of which about 15-20% useful (1). The plate was, in turn, Pb 1, 2, 3, 4 cm thick and Fe 4 cm thick.

The experiment was performed in Arcetri at 120 m a. s.l. under a roof of about 150 g/cm² (masonry).

3. – Angles of the electrons relative to the shower axis.

It would surely have been preferable to study « true » angles, thus eliminating an « instrumental » source of fluctuations, but the amount of work would have been prohibitive. Thus in what follows we shall always refer to « projected » angles. The stereoscopic pair has been utilized, however, not only in cases of interpretation, but to establish the point where the shower leaves the plate. We have classified as photon-initiated only those showers without visible primary whose origin lays in that region of the plate which contains the showers having visible primaries.

Whenever there is a visible primary, its direction is taken as the shower axis. Two methods have been tried to fix the direction of the axis when no

(1) The controlling was made loose on purpose. That has not avoided, however, the necessity of correcting the results from the selectivity effects of the apparatus. (See n. 4).

primary is visible. The first one is to associate a unitary vector to each track and find the resultant. The second one consists in taking a weighted average of the angles of the tracks (referred to an arbitrary direction), the weight being chosen inversely proportional to the (angular) distance of the track to the nearest one (²). Probably, this second method, though more laborious, is more accurate.

For a test, these two methods have been applied to showers having visible primaries. Fig. 2 shows the histograms of the differencies between calculated axes and the corresponding primaries. The method of vectors (full line) gives a standard deviation of 7.1°, that of weighted averaged (broken line) 5.8°. (The curve is a gaussian with standard deviation 6.5°).

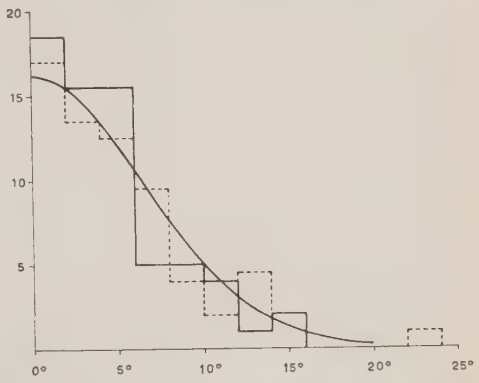

Fig. 2. – Histogram of the differencies between the directions of the (ionizing) primaries and the corresponding axes of the showers determined from the tracks of the secondaries according to two different methods.

Indeed, the uncertainty in the position of the axis affects the angular distribution and we shall give here a method to find out the true distribution. Its application in the present instance is however strongly limited by the statistical imprecision of the data.

Let $\varphi(\theta)$ be the true distribution, where θ is the angle (reckoned always positively) that the electron makes with the shower axis. It is easy to see that, were the axis always displaced by the same amount s, one should find a distribution $r(s, \theta)$ given by

$$r(s, \theta) = \tfrac{1}{2}\left[\varphi(\theta + s) + \varphi(s - \theta)\right] \qquad \theta < s\,,$$

$$r(s, \theta) = \tfrac{1}{2}\left[\varphi(\theta + s) + \varphi(\theta - s)\right] \qquad \theta > s\,.$$

If the values of s are distributed according to a law $A(s)$, the experimentally found distribution would turn out to be

$$F(\theta) = \int A(s)\, r(s, \theta)\, \mathrm{d}s, \tag{1}$$

where $r(s, \theta)$ is related to $\varphi(\theta)$ by the above relations.

(²) To avoid giving a large weight to tracks which are near only on perspective grounds, their vicinity has been checked on the twin picture, choosing the lowest weight.

It is not difficult to solve this equation if one assumes $A(s)$ to be rectangular, that is

$$A(s) = 1/\bar{s} \qquad 0 < s < \bar{s}\,,$$

$$A(s) = 0 \qquad \text{everywhere else}\,.$$

The preceding equations then give
for $\theta < \bar{s}$

$$2\bar{s}F(\bar{s},\theta) = \int_0^{\bar{s}} \varphi(\theta + s)\,ds + \int_0^{\theta} \varphi(\theta - s)\,ds + \int_{\theta}^{\bar{s}} \varphi(s - \theta)\,ds\,, \tag{2}$$

and for $\theta > \bar{s}$

$$2\bar{s}F(\bar{s},\theta) = \int_0^{\bar{s}} \varphi(\theta + s)\,ds + \int_0^{\bar{s}} \varphi(\theta - s)\,ds\,. \tag{2'}$$

($F(\bar{s},\theta)$ has been written in place of $F(\theta)$ to remind that the distribution function F depends on $\bar{s}$).

By simply taking derivatives, we can eliminate the integral signs and get

$$2\bar{s}\frac{\partial F}{\partial \theta} = \varphi(\bar{s} + \theta) - \varphi(\bar{s} - \theta) \qquad \theta < \bar{s}\,, \tag{3}$$

$$2\bar{s}\frac{\partial F}{\partial \theta} = \varphi(\theta + \bar{s}) - \varphi(\theta - \bar{s}) \qquad \theta > \bar{s}\,. \tag{3'}$$

These equations can easily be solved in the assumption that φ goes asymptotically to zero when $\theta \to \infty$ (as is practically the case here). Indeed, by putting

$$\theta - \bar{s} = \theta'\,,$$

we get from (3') (F' means $\partial F/\partial\theta$ considered as a function of θ)

$$\varphi(\theta') = \varphi(\theta' + 2\bar{s}) - 2\bar{s}F'(\theta' + \bar{s})\,, \tag{I}$$

and thence, adding repeatedly $2\bar{s}$ to the argument θ'

$$\varphi(\theta' + 2\bar{s}) = \varphi(\theta' + 4\bar{s}) - 2\bar{s}F'(\theta' + 3\bar{s})\,, \tag{II}$$

$$\varphi(\theta' + 4\bar{s}) = \varphi(\theta' + 6\bar{s}) - 2\bar{s}F'(\theta' + 5\bar{s})\,, \tag{III}$$

. . . . . .

Adding these equations (and now dropping the prime for θ) we have

$$\varphi(\theta) = -2\bar{s}\sum_{r=0}^{\infty} F'\left[\theta + (2r+1)\bar{s}\right] \qquad \theta > \bar{s}\,. \tag{4}$$

To find $\varphi(\theta)$ in the region $0 < \theta < s$, one can start from (3), which can be written

$$2\bar{s}F'(\theta'') = \varphi(\bar{s} + \theta'') - \varphi(\bar{s} - \theta'').$$

By putting

$$\theta'' = \bar{s} - \theta\,,$$

we get

$$\varphi(\theta) = \varphi(2\bar{s} - \theta) - 2\bar{s}F'(\bar{s} - \theta)\,.$$

As $2s - \theta$ is larger than $\bar{s}$, (4) can be applied to the right hand side, and this gives

$$\varphi(\theta) = -2\bar{s}\sum_{r=0}^{\infty} F'\left[(2r+1)\bar{s} - \theta\right] \qquad \theta < \bar{s}\,. \tag{4'}$$

If $\varphi(\theta)$ goes to ∞ when $\theta \to 0$ (as the theory would require in our case) it is convenient to have a formula giving

$$\overline{\varphi}_{0,\theta_0} = \frac{1}{\theta_0}\int_0^{\theta_0} \varphi(\theta)\,\mathrm{d}\theta\,,$$

for small values of θ_0. This formula is very simple if one takes $\theta_0 = \bar{s}$, and can easily be deduced from (4') which gives

$$\overline{\varphi}_{0,\bar{s}} = 2\left[F(0) + F(2\bar{s}) + F(4\bar{s}) + \ldots - F(\bar{s}) - F(3\bar{s}) - F(5\bar{s})\ldots\right]. \tag{5}$$

For small θ, a power law may be used for $\varphi(\theta)$:

$$\varphi(\theta) = a\theta^{-\nu} \qquad (\theta \to 0,\ \nu < 1). \tag{6}$$

Equations (2) and (2') then give

$$F(\theta) = \frac{a}{2\bar{s}(1-\nu)}\left[(\theta + \bar{s})^{1-\nu} + (\bar{s} - \theta)^{1-\nu}\right] \qquad \theta < \bar{s}\,, \tag{7}$$

and

$$F(\theta) = \frac{a}{2\bar{s}(1-\nu)}\left[(\theta + \bar{s})^{1-\nu} - (\theta - \bar{s})^{1-\nu}\right] \qquad \theta > \bar{s}\,. \tag{7'}$$

By simply looking at derivatives, one finds that $F(\theta)$ is a bell shaped curve (the ordinate axis being the simmetry one) with an inflection point at $\theta = \bar{s}$ where, for positive ν, one would have (presumably as a consequence of the discontinuity introduced in $A(s)$): $\partial F/\partial\theta = -\infty$ (Eq.'s (3) and (3′)). From an elaboration of the results of ROBERG and NORDHEIM [3] which was made by S. FRANCHETTI and M. GIOVANNOZZI [4] to cover the case of Pb, a value 1/3 for ν has been found for $\theta < \sim 15^{\circ}$.

4. – Recording Probabilities.

This is a point which, especially in the past, has received too little attention. It is however essential if one is to get objective results from the raw experimental material. In the present case it does affect, though not very much, angular distributions but it proves decisive for distributions according to the number of electrons.

The probability of getting a coincidence of a given order when n particles fall on a group of counters, has been calculated in the case of equal counters put side by side and no particles falling outside the counters, by M. DELLA CORTE [5] and E. SCHRÖDINGER [6]. Here the problem is somewhat different, since the area of the counters is but a fraction of that on which the shower particles are spread. Essentially the recording will be determined by the two lower counters (see geometry of the apparatus in fig. 1): we have therefore to evaluate the probability P that a shower of n particles will start a coincidence, each particle having an average probability σ of hitting one of the two counters. Simple considerations give the formula

$$P_{n,1} = \binom{n}{x} \sigma^x (1-\sigma)^{n-x}\,, \tag{8}$$

for the probability that a counter will be hit by only x ($\leqslant n$) particles. From this, taking every possible combination that gives a coincidence, the following formula for P can easily be derived

$$\begin{aligned} P = {} & P_{n,1}\left[P'_{n-1,1} + P'_{n-1,2} + \ldots + P'_{n-1,n-1}\right] + \\ & + P_{n,2}\left[P'_{n-2,1} + P'_{n-2,2} + \ldots + P'_{n-2,n-2}\right] + \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & + P_{n,n-2}\left[P'_{2,1} + P'_{2,2}\right] + \\ & + P_{n,n-1}P'_{1,1}\,. \end{aligned} \tag{9}$$

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[3] J. ROBERG and L. W. NORDHEIM: *Phys. Rev.*, **75**, 444 (1949).
[4] S. FRANCHETTI and M. GIOVANNOZZI: *Nuovo Cimento*, **8**, 312 (1951).
[5] M. DELLA CORTE: *Rend. Acc. Lincei*, VIII, **2**, 311 (1947).
[6] E. SCHRÖDINGER: *Proc. Phys. Soc.*, **64**, 1040 (1951).

Here $P'_{h,k}$ means the same as $P_{h,k}$ except that it must be calculated replacing σ by $\sigma' = \sigma/(1-\sigma)$.

The determination of σ is a rather delicate matter. For the sake of simplicity let us consider first the n particles to be distributed uniformly randomly inside a cone having semiaperture α; σ is then a function of α and of the position of the cone relative to the counters. The pictures show that for at least 70% of the showers the axis makes an angle $\leqslant 12°$ with the vertical. Within these limits average values have been calculated for $\sigma(\alpha)$ with the results shown in Table I.

TABLE I.

α	σ	α	σ
6°	0.11	20°,5	0.125
11°	0.16	24°	0.098
13°	0.24	32°	0.060
17°	0.165	40°	0.034

By means of these values and of formula (9), the curves in fig. 3 giving the inverse $1/P$ of the recording probabilities as a function of n and α have been calculated.

With the aid of these curves a weight can be attached to each shower. Of course, some attention is required to take into account the fact that the distribution of the electrons is far from uniform. It has been tried to allow for this circumstance by taking as weight the lowest value of $1/P$ among those which result from considering a given shower as composed by n particles in α, or $n-1$ in α' ($<\alpha$) and so on. For showers of only two electrons the semiaperture α has been taken equal to the angle between the electrons (not half the angle). Moreover, since

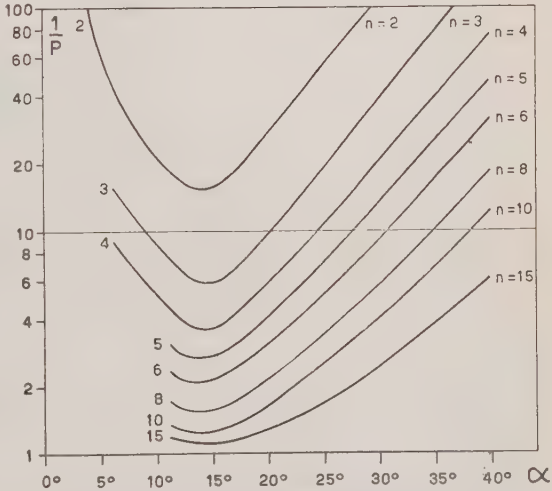

Fig. 3. – The inverse $1/P$ of the probability of recording a shower supposed to consist of n (ionizing) particles uniformly distributed in a cone of semiaperture α.

α has to be a « true » angle, the projected angles measured on the pictures have been multiplied by $\pi/2$ to take account on the average, though rather crudely, of the reduction of angles by projection (7).

Sometimes there is evidence that the recording of the shower is due to particles not belonging to it. In that case the shower has been given the weight of those with same n which appear to be independent.

5. – Angular Distributions.

A preliminary question is whether the angular distributions be dependent on the nature of the primary and the number n of the electrons. There is no evidence of a dependence of the former kind, while there is clearly one of the latter.

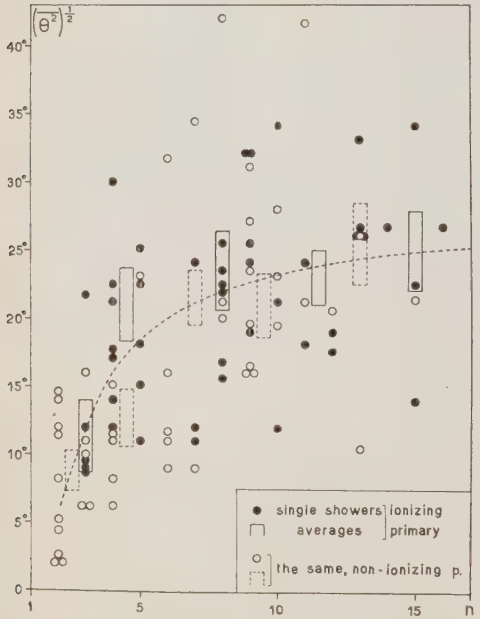

Fig. 4. – The root mean square (projected) angle of the electrons in a shower as a function of the number of electrons.

As the data are too few to build up separate distributions, we must resort to averages to get an idea of the greater or smaller spreading of the showers. Fig. 4 shows the root mean square angles (relative to the shower axis), i.e.

$$\left(\frac{1}{n}\sum_{i=1}^{n}\theta_i^2\right)^{1/2}.$$

The small circles refer to single showers, while the rectangles give averages with their statistical errors. We see that the spreading increases with increasing n, tending apparently to a limit. The dashed curve has the (empirical) equation

$$y = 28^\circ - \frac{48^\circ}{n},$$

and is drawn to show this saturation character.

(7) This consideration does not apply for $n = 2$ and angles larger than that subtended by the counters (about (10°): in this case, indeed, the pictures give about the true angle.

The average aperture of the showers, calculated from the average (experimental) weights by using the curves in fig. 3 shows the same trend, though less marked. This is shown in Table II.

Table II.

n	Average weight	« True » angle according to the curves fig. 3	Corresponding mean projected angle
2	40	23°	14°.5
3–4	14	25°	16°
5–6	7.2	27°	17°
7–9	6.0	30°	19°
10–12	4.5	32°	20°
13–15	3.5	33°	21°
16–18	3.0	35°	22°

For showers of at least 5 electrons, an angular distribution has been derived assigning the tracks to 6 angular classes and giving them the same weight of the shower to which they belong. The results are collected in Table III together with the effective numbers (not weighted). (The errors given are standard deviations).

Table III.

Class	*Number of tracks per* 10°					
	A) Ioniz. primary		*B*) Non-ioniz. primary		*C*) Uncertain	
	effect.	correct.	effect.	correct.	effect.	correct.
0°–5°	54	49.2 ± 5.2	52.4	47.8 ± 6	63.2	57.4 ± 6.3
5°–15°	31.5	32.6 ∓ 3	34.3	32.2 ± 3.5	33.6	33.6 ± 4
15°–30°	15.6	14.6 ± 1.7	14.0	14.7 ± 2	12.7	13.3 ± 2
30°–50°	6.2	7.3 ± 1	6.4	7.4 ± 1.3	6.0	6.1 ± 1
50°–70°	2.1	2.6 ± 0.7	2.2	2.9 ± 1	1.6	2.2 ± 1
70°–90°	0.5	0.5 ± 0.3	0.6	0.6 ± 0.4	0.2	0.2 ± 0.2

Since no significant differences are observed between the various categories of showers, those whose axes have been calculated, i.e. the categories *B*) and *C*) of Table III have been put together to give the results shown in fig. 5.

The curve is an example of a curve *compatible* with the experimental data and the points marked by triangles have been obtained from it using the procedure explained in n. 3. We see that the corrections are small (except, obviously, for $\theta \to 0$).

Fig. 6 shows the comparison with the curves calculated by S. FRANCHETTI and M. GIOVANNOZZI (8), starting from ROBERG and NORDHEIM's results (9). The full and the broken curves represent respectively the results corrected and uncorrected for the larger absorption suffered by the more inclined electrons. The experimental points include all the showers (with $n \geqslant 5$). The point at 2,5° is the average of the mean values in the interval 0°-5°. (For categories B) and C) an evaluation has been attempted by means of eq. (5)). The theoretical curves have been normalized at this point.

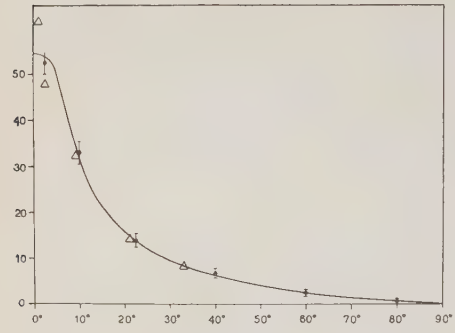

Fig. 5. – Angular distribution for photon-initiated showers. (Projected angles)

As can be seen, there is fair agreement with the full curve. The lower values of the experimental points in the central region may however be significant. In fact the root mean square value calculated from the full curve is 27° while the experimental value is only 23°. On the other hand, a limiting value of 28°, practically coincident with the theoretical one is indicated by the dashed curve in fig. 4. This seems to indicate that the data of ROBERG and NORDHEIM refer to a limiting distribution which is approached by increasing n.

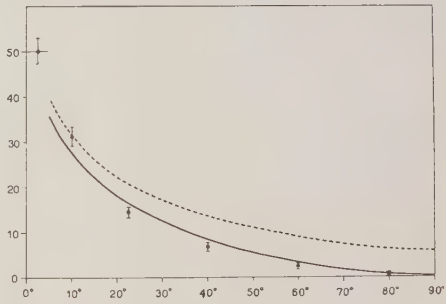

Fig. 6. – Angular distribution for showers of either origin. Curves are theoretical distributions: broken line, uncorrected and full line, corrected for the larger absorption suffered by inclined electrons. (projected angle)

6. – Distribution according to the number of electrons.

Table IV shows as a function of n the sums of the weights of the showers with a given n, and the corresponding effectively observed numbers (in brackets). Of course the former are supposed to represent the true proportion of the showers.

(8) Loc. cit. ref. (4).
(9) Loc. cit. ref. (3).

TABLE IV.

n	Ioniz. prim.		Non-ioniz. prim.		Uncertains		Sums	Averages
2	0		455	(10)	305	(9)	760	760 ± 180
3	76	(6)	74	(4)	112	(8)	260	244 ± 41
4	111	(8)	75	(5)	42	(4)	228	
5	54	(6)	17	(1)	48	(6)	119	90 ± 18
6	0		19	(4)	41	(8)	60	
7	14	(3)	10	(2)	23	(3)	47	
8	26	(5)	24	(3)	0		50	60 ± 11
9	29	(4)	35	(6)	17	(3)	81	
10	11	(3)	8	(2)	14	(3)	33	
11	7	(2)	11	(2)	15	(1)	33	26 ± 6.5
12	7	(2)	5	(1)	0		12	
13	14	(4)	8.5	(2)	8.5	(2)	31	
14	3	(1)	0		0		3	14 ± 4
15	5.5	(2)	2.5	(1)	0		8	
16	5.5	(2)	0		4	(1)	9.5	
17	0		0		3.2	(1)	3.2	5 ± 2.2
18	0		0		2.5	(1)	2.5	
>18	(altogether 8 showers)							

No significant differencies seem to exist between electron initiated and photon-initiated showers, except that there are no showers of the former tipe with $n = 2$. Collecting all the showers together, as has been done in columns 5 and 6, and plotting the data logarithmically we obtain the diagram in fig. 7, which shows a very good agreement with a law *const.* n^{-2}, except for the last datum which suggests a steeper decrease for high values of n. This point is confirmed by the small number of showers with $n > 18$. Assuming a power law in this region too, and making use of the approximation formula

$$\varrho = \sum_{n=n_0}^{\infty} Cn^{-x} = \frac{1}{2} Cn_0^{-x} + \int_{n_0}^{\infty} Cn^{-x}\,\mathrm{d}n = Cn_0^{-x}\left(\frac{1}{2} + \frac{n_0}{x-1}\right),$$

with $n_0 = 16$, $Cn_0^{-x} = 8 \pm 3$, $\varrho = 39 \pm 11$, we get [10]

$$x = 4.6 \pm 1.9\,.$$

[10] To evaluate ϱ one must know, beside the number of the showers, the average weight to be attributed to those with $n > 18$. A glance at the values in Table II shows however that such an average weight cannot be far from 3.

Since when n is greater than, say, 10, it should be proportional to the energy of the primary, the exponent x should coincide with that of the (differential) spectrum of the primary electrons or photons in the high energy region. The value commonly accepted $x = \sim 3$ lies within the limits of error.

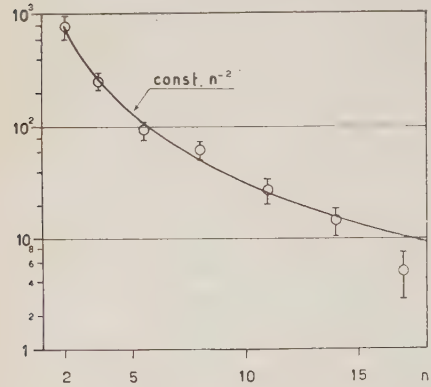

Fig. 7. – Relative frequencies of showers according to the number of electrons in them. Point at $n = 2$ refers to photon initiated only, others to showers of either origin together.

7. – Ratio of non-ionizing to ionizing primaries.

The existence of numerous showers of photonic origin with $n = 2$, is an important element. Without these, i.e. for $n \geqslant 3$, the number of showers of either origin is about the same. For $n \geqslant 2$ we get instead, because of the notable weight of those with $n = 2$ (data of Table III)

$$R = \frac{\text{n. showers with phot. primary}}{\text{n. showers with electr. primary}} = 2 \pm 0.4 .$$

This result appears to contradict some classical experiment on the subject, such as that of REGENER ([11]). Ther are however at least two good reasons for this disagreement, namely: *a*) the low probability (of which allowance was not made) of recording showers of only two electrons by means of Regener's apparatus (whose geometry was very similar to our own) and, *b*) the circumstance, already emphasized by DAUDIN and TSAI-CHU ([12]), that showers are very often accompanied by associated ionizing particles which can trigger the counters even in the absence of any ionizing primary.

At 2 cm Pb, Regener's data are: 60 coinc./hr for the total frequency of showers and 20 coinc./hr for those classified as being of photonic origin. On the basis of *a*) alone, Regener's data should give something like the ratio of our uncorrected results, i.e. about 1 : 1. Should the departure from this value be attributed entirely to circumstance *b*), this would require a value 1/3 for the fraction of photonic showers accompanied by ionizing particles. Such a

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([11]) V. H. REGENER: *Ric. Scient.*, **11**, 66 (1940), and also: B. ROSSI: *High Energy Particles* (New York, 1952), p. 328.

([12]) J. DAUDIN and TSAI-CHU: *Compt. Rend. Ac. Scienc.*, **234**, 1277 (1952).

value is not at all inconsistent with our observations and those of DAUDIN and TSAI-CHU. However, a closer comparison is prevented by the differencies in experimental conditions.

To get a clear interpretation of Regener's experiment (beside our own) it must be added that the contribution due to mesons accompanied by a secondary is practically negligible (and the more so if one does not correct for the low recording probability). In fact the examination of the pictures shows hardly some such event. Comparison cannot be made with Schwegler's experiment ([13]), where the geometry is altogether different and so presumably the recording probability.

It must also be noted that, arising as it does from a contribution due to showers of only two electrons, the excess of the photonic primaries over the electronic ones appears as a feature of the very lowest energy part of the cosmic radiation ([14]). This is in qualitative agreement with a series of experiments on very soft cosmic rays performed in 1939 by G. BERNARDINI and B. FERRETTI ([15]).

It is a pleasure for the writer to thank Prof. M. DELLA CORTE, Dr. M. GIOVANNOZZI and Miss M. C. AMERIGHI for the share they took in collecting the data, as well as G. TORTORICI for the drawing of the graphs.

([13]) A. SCHWEGLER: *Zeits. f. Phys.*, **96**, 62 (1935), and also: D. J. X. MONTGOMERY: *Cosmic Ray Physics*, (Princeton, 1949), p. 268.

([14]) The cut-off due to the chamber and counter walls amounts to about 3 MeV.

([15]) G. BERNARDINI and B. FERRETTI: *Ric. Scient.*, **10**, 39 (1939).

RIASSUNTO

In questa nota si descrive il dispositivo sperimentale e i metodi adoperati per l'analisi dei dati che sono stati raccolti fotografando sciami elettronici con una camera di Wilson controllata « dal basso ». Particolarmente importante si dimostra la correzione dell'effetto di selettività del dispositivo di segnalazione. Si discutono i resultati ottenuti con uno dei vari setti adoperati nell'interno della camera (2 cm Pb). Questi riguardano la ripartizione angolare degli elettroni nei singoli sciami, la distribuzione di questi secondo il numero degli elettroni e il rapporto fra il numero degli sciami fotonici e quello degli sciami elettronici.

Non-Adiabatic Treatment of Nucleon Pion Scattering.

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(ricevuto il 18 Gennaio 1953)

Summary. — The Tamm-Dancoff non adiabatic method is applied to the study of nucleon-pion scattering. An integral equation is obtained. It is resolved, making use of some approximations, getting numerical results in satisfactory agreement with experiment. The existence of an isobaric state of the nucleon with spin 3/2 and isotopic spin 3/2 is also deduced.

1. – Introduction.

The pseudoscalar meson theory has been recently applied [1] to the investigation of the nucleon-pion scattering. However, these calculations, which make use of the perturbation method, are not in good agreement with experiments [2].

On the other hand Brueckner [3] has given a phenomenological explanation of the experimental results assuming the existence of a nucleon isobar with spin 3/2 and isotopic spin 3/2.

In this paper we shall treat the nucleon-pion system using the Tamm-Dancoff method [4] which has already been applied successfully by Levy [5]

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[1] J. Ashkin, A. Simon and R. E. Marshak: *Progr. Theor. Phys.*, **5**, 634 (1950); M. Cini and L. A. Radicati: *Nuovo Cimento*, **8**, 317 (1951); M. Peshkin: *Phys. Rev.*, **81**, 425 (1951).

[2] H. L. Anderson *et al.*: *Phys. Rev.*, **85**, 936 (1952); **86**, 793 (1952).

[3] K. Brueckner: *Phys. Rev.*, **86**, 106 (1952); K. Brueckner and K. M. Watson: *Phys. Rev.*, **86**, 923 (1952); G. Wentzel: *Phys. Rev.*, **86**, 437 (1952).

[4] I. Tamm: *Journ. Phys. (U.S.S.R.)*, **9**, 449 (1945); S. M. Dancoff: *Phys. Rev.*, **78**, 382 (1950); D. Baroncini: *Nuovo Cimento*, **9**, 642 (1952).

[5] M. Levy: *Phys. Rev.*, **88**, 72, 724 (1952).

to the low energy proton-neutron system. With this method we shall take into account, with a single integral equation, of all Feynmann graphs which have no more than one pair and no more than two mesons at one time.

Clearly, if an iteration procedure is applied to our integral equation, the results will be the same as those of the perturbation method. On the other hand the existence of a resonance seems to indicate that, at least for low energies the iteration method does not work (6). Therefore we shall use the more general methods of Schmidt and Fredholm. Both these methods give the solution of an inhomogeneus integral equation as a fraction whose denominator depends, in our case, only on the coupling constant and on the total energy of the system.

Choosing the value of the coupling constant which explains the low energy proton neutron system, we find that the complex values of the energy for which the denominator vanishes correspond to resonances: the real part representing the resonance energy, the imaginary one the resonance width.

2. – The integral equation of the problem.

We shall assume the pion nucleon interaction to be pseudoscalar and charge symmetrical (7):

$$H' = ig \sum_{r=1}^{3} (\psi^* \beta \gamma^5 \tau^r \psi) \Phi^r, \tag{1}$$

ψ is the nucleon wave field, Φ the meson field; β, γ^5 are the well known Dirac matrices; τ^r are the three isotopic spin matrices, and the sum is over the three charge states of the pion.

The Schrödinger equation for the wave functional Ψ of the system is as follows:

$$(H_0 + H')\Psi = W\Psi\,, \tag{2}$$

where H_0 is the free Hamiltonian and W the eigenvalue of the total energy.

Let Ψ be expanded in a series of eigenfunctions of H_0: In our case this development will contain states with *one* nucleon, m mesons, n nucleon pairs:

$$\Psi = \sum_{\lambda, m, n} a_\lambda^{(m,n)} |\lambda, m, n\rangle\,, \tag{3}$$

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(6) S. D. Drell and E. M. Henley: *Phys. Rev.*, **88**, 1053 (1950).

(7) A system of units in which $h = c = 1$ is used in this paper.

where λ is a variable which specifies the momenta, spins, etc., of the system and $a_\lambda^{(m,n)}$ the probability amplitude of the state $|\lambda, m, n\rangle$. From (2) we obtain the following set of integral equations:

$$(4) \qquad [W - E_\lambda^{(m,n)}] a_\lambda^{(m,n)} = \sum_{q=n-1}^{n+1} \sum_{p=m\pm 1} \sum_{\mu} \langle \lambda, m, n | H' | \mu, p, q \rangle a_\mu^{(p,q)} .$$

In the following we shall consider only the effect of states with no more than two mesons and no more than one pair. In this approximation all the amplitudes can be expressed as functions of $a_\lambda^{(1,0)}$. Therefore, in the center of mass system we get the following equation:

$$(5) \qquad (W - E_p - \omega_p) a_r(\boldsymbol{p}) = \frac{g^2}{8\pi} \sum_s \int [K_1(\boldsymbol{p}, \boldsymbol{q}) \tau^s \tau^r + K_2(\boldsymbol{p}, \boldsymbol{q}) \tau^r \tau^s] a_s(\boldsymbol{q}) \, \mathrm{d}^3 q ,$$

where: $E_p = \sqrt{p^2 + M^2}$; $\omega_p = \sqrt{p^2 + \mu^2}$ (M= nucleon mass; μ = pion mass) and the kernels $K_1(\boldsymbol{p}, \boldsymbol{q})$ and $K_2(\boldsymbol{p}, \boldsymbol{q})$ are given by:

$$(6a) \qquad K_1(\boldsymbol{p}, \boldsymbol{q}) = \frac{1}{2\pi^2 \sqrt{\omega_p \omega_q}} \left[- \frac{\langle \boldsymbol{p}^+ | i\beta\gamma^5 | \boldsymbol{p} + \boldsymbol{q}^- \rangle \langle \boldsymbol{p} + \boldsymbol{q}^- | i\beta\gamma^5 | \boldsymbol{q}^+ \rangle}{W - E_p - E_q - E_{p+q}} + \right.$$

$$\left. + \frac{\langle \boldsymbol{p}^+ | i\beta\gamma^5 | \boldsymbol{p} + \boldsymbol{q}^+ \rangle \langle \boldsymbol{p} + \boldsymbol{q}^+ | i\beta\gamma^5 | \boldsymbol{q}^+ \rangle}{W - \omega_p - \omega_q - E_{p+q}} \right] ,$$

$$(6b) \qquad K_2(\boldsymbol{p}, \boldsymbol{q}) = \frac{1}{2\pi^2 \sqrt{\omega_p \omega_q}} \left[- \frac{\langle \boldsymbol{p}^+ | i\beta\gamma^5 | - \rangle \langle - | i\beta\gamma^5 | \boldsymbol{q}^+ \rangle}{W - E_p - E_q - M - \omega_p - \omega_q} + \right.$$

$$\left. + \frac{\langle \boldsymbol{p}^+ | i\beta\gamma^5 | + \rangle \langle + | i\beta\gamma^5 | \boldsymbol{q}^+ \rangle}{W - M} \right] ,$$

where $|\boldsymbol{p}^+\rangle$, $|\boldsymbol{p}^-\rangle$ represent nucleon states with momentum $\boldsymbol{p}$ and energy positive and negative respectively (we shall write simply $|+\rangle$ and $|-\rangle$ when $\boldsymbol{p} = 0$).

Separating the isotopic spin variables (see Appendix I) we obtain the following equations for isotopic spins 3/2 and 1/2:

$$(7a) \qquad (W - E_p - \omega_p) a_{3/2}(\boldsymbol{p}) = \frac{2g^2}{8\pi} \int K_1(\boldsymbol{p}, \boldsymbol{q}) a_{3/2}(\boldsymbol{q}) \, \mathrm{d}^3 q ,$$

$$(7b) \qquad (W - E_p - \omega_p) a_{1/2}(\boldsymbol{p}) = \frac{2g^2}{8\pi} \int [3K_2(\boldsymbol{p}, \boldsymbol{q}) - K_1(\boldsymbol{p}, \boldsymbol{q})] a_{1/2}(\boldsymbol{q}) \, \mathrm{d}^3 q .$$

The kernel $K_2(\boldsymbol{p}, \boldsymbol{q})$ includes self-energy effects and therefore gives rise to renormalization difficulties. In this paper we shall treat only the $T = 3/2$

scattering, where the kernel $K_2(\boldsymbol{p}, \boldsymbol{q})$ does not appear. The $T = 1/2$ scattering will be the object of a further investigation.

The kernel $K_1(\boldsymbol{p}, \boldsymbol{q})$ may be simply evaluated, using the formula:

$$\left| \boldsymbol{p}^{\pm} \right\rangle = \frac{1}{D_p} \Lambda_{\boldsymbol{p}}^{\pm} \left| \pm \right\rangle , \tag{8}$$

where:

$$D_p = \sqrt{\frac{2E_p}{M + E_p}} , \qquad \Lambda_{\boldsymbol{p}}^{\pm} = \frac{1}{M + E_p} \left[E_p \pm \beta M \pm (\boldsymbol{\alpha} \cdot \boldsymbol{p}) \right] .$$

Thus we obtain:

$$K_1(\boldsymbol{p}, \boldsymbol{q}) = \tag{9}$$

$$= \frac{1}{2\pi^2 \sqrt{D_p D_q \omega_p \omega_q}} \left[\frac{(E_{p+q} + M) + (E_p + E_q - 2M) + \dfrac{(E_{p+q} + E_p + E_q + M)(\boldsymbol{\sigma p})(\boldsymbol{\sigma q})}{(E_p + M)(E_q + M)}}{2E_{p+q}(W - E_p - E_q - E_{p+q})} - \right.$$

$$\left. - \frac{(E_{p+q} - M) - (E_p + E_q - 2M) + \dfrac{(E_{p+q} - E_p - E_q - M)(\boldsymbol{\sigma p})(\boldsymbol{\sigma q})}{(E_p + M)(E_q + M)}}{2E_{p+q}(W - \omega_p - \omega_q - E_{p+q})} \right] .$$

Since the kernel $K_1(\boldsymbol{p}, \boldsymbol{q})$ becomes more and more irrelevant as p and q increase beyond M, we shall limit the region of integration to the interval from 0 to ξM, where ξ is a number of the order of unity whose exact value will be chosen so as to fit in the best way with the experimental data. For this range of p and q values the nucleon recoil is small. Therefore the kernel may be written in the form:

$$K_1(\boldsymbol{p}, \boldsymbol{q}) = A(p, q) + B(p, q)(\boldsymbol{\sigma p})(\boldsymbol{\sigma q}) + C(pq)(\boldsymbol{pq}) , \tag{10}$$

where:

$$A(p, q) = \frac{-1}{2\pi^2 \sqrt{\omega_p \omega_q}(W - 3M)} , \tag{11a}$$

$$B(p, q) = \frac{1}{2\pi^2 \sqrt{\omega_p \omega_q}} \left[\frac{-1}{4M^2(W - M - \omega_p - \omega_q)} + \frac{1}{2M^2(W - 3M)} \right] , \tag{11b}$$

$$C(p, q) = \frac{1}{2\pi^2 \sqrt{\omega_p \omega_q}} \left[\frac{1}{2M^2(W - M - \omega_p - \omega_q)} + \frac{1}{2M^2(W - 3M)} \right] . \tag{11c}$$

Developing the probability amplitude $a_{3/2}(p)$ in (7*a*) as a series of eigenfunctions of the total angular momentum j and of the orbital angular momentum l, we get three integral equations for the s, $p_{3/2}$, $p_{1/2}$ waves respectively:

$$(W - E_p - \omega_p)\varphi_s(p) = g^2 \int A(p, q)\varphi_s(q)q^2 \,\mathrm{d}q \,, \tag{12a}$$

$$(W - E_p - \omega_p)\varphi_{p_{3/2}}(p) = \frac{g^2}{3}\int C(p, q)\,\varphi_{p_{3/2}}(q)pq^3 \,\mathrm{d}q \,, \tag{12c}$$

$$(W - E_p - \omega_p)\varphi_{p_{1/2}}(p) = g^2 \int \left[B(p, q) + \frac{1}{3}\, C(p, q)\right] \varphi_{p_{1/2}}(q)pq^3 \,\mathrm{d}q \,, \tag{12c}$$

Let k be the value of p for which $W - E_p - \omega_p$ vanishes, then equations (12) may be written in the form:

$$\varphi_s(p) = \delta(p - k) - g^2 \int \frac{A(p, q)}{\omega_p - \omega_k}\, \varphi_s(q)q^2 \,\mathrm{d}q \,, \tag{13a}$$

$$\varphi_{p_{3/2}}(p) = \delta(p - k) - \frac{g^2}{3}\int \frac{C(p, q)}{\omega_p - \omega_k}\, \varphi_{p_{3/2}}(q)pq^3 \,\mathrm{d}q \,, \tag{13b}$$

$$\varphi_{p_{1/2}}(p) = \delta(p - k) - g^2 \int \frac{B(p, q) + \dfrac{C(p, q)}{3}}{\omega_p - \omega_k}\, \varphi_{p_{1/2}}(q)pq^3 \,\mathrm{d}q \,. \tag{13c}$$

Both the Schmidt and the Fredholm method (see Appendix II) give the solution of the integral equations in the form:

$$\varphi(p) = \delta(p - k) + \int \frac{\Delta(k; p, q)}{D(k)}\, \delta(p - k)\,\mathrm{d}k = \delta(p - k) + \frac{\Delta(k; p, k)}{D(k)} \,. \tag{14}$$

The phase shifts are given by the formula:

$$\operatorname{tg}\eta = -\pi \frac{1}{D(k)} \lim_{p \to k} [(p - k)\Delta(k; p, k)]. \tag{15}$$

There is a resonance for those complex values of k for whick $D(k)$ vanishes. Let $a + ib$ be one of these values. In the neighbourhood of a $\operatorname{tg}\eta$ may be approximated by:

$$\operatorname{tg}\eta = \frac{C}{b^2 + (\omega_k - a)^2} \,, \tag{16}$$

(16) is identical with the Breit-Wigner formula [8]: a is the resonance energy, b/a the resonance width.

3. – Numerical results.

$p_{3/2}$ *wave.* – The potential is attractive. Therefore the phase shift will be greater than the one given by the perturbation method. Let us evaluate the denominator $D(k)$ by the Fredholm method:

$$D(k) = 1 - \frac{g^2}{3}\int \frac{C(k;\, p,\, q)}{\omega_k - \omega_p}\, \mathrm{d}p \,. \tag{17}$$

The second term in the right hand side has a maximum for a value of k of the order of $0.4M$, which is only weakly affected by the choice of ξ. On the other hand, since the value of the maximum is approximately one, $D(k)$ has a minimum which is very sensitive to the values of ξ and g.

With $g^2/4\pi = 10.5$ (in accord with the results of LEVY) and $\xi = 0.92$, we may write for $D(k)$ in the neighbourhood of resonance:

$$D(k) = 0{,}07 + 6{,}5\left(\frac{\omega_k}{M} - 0{,}35\right)^2. \tag{18}$$

For the resonance energy and width we find thus $0.35M$ and 0.30 respectively, as compared with $0.3M$ and 0.23 of BRUECKNER. The phase shift for 135 and 110 MeV become 38° and 26° (see table I).

$p_{1/2}$ *wave.* – Small repulsive potential. Using the Fredholm method one obtains $-3°$ and $-2°$ respectively for the phase shifts at 135 MeV and 110 MeV.

s *wave.* – The potential is repulsive and very large. The phase shift is strongly decreased as confronted with the one given by the perturbation method. The Schmidt method is easily applicable. With the same values of g and ξ already employed for the $p_{3/2}$ wave we obtain $-20°$ and $-17°$ for the phase shifts at 135 and 110 MeV.

Our numerical results are summarized in the following table.

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[8] See e.g. J. M. BLATT and V. F. WEISSKOPF: *Theoretical Nuclear Physics* (New York, 1952), p. 391.

TABLE I.

	135 Mev		110 Mev	
	Theoretical	Experimental	Theoretical	Experimental
s	-20°	-25°	-17°	-15°
$p_{3/2}$	$+38^\circ$	$+35^\circ$	$+26^\circ$	$+25^\circ$
$p_{1/2}$	-3°	$+10^\circ$	-2°	0°

One sees that there is a satisfactory agreement between the pseudoscalar meson theory and the experimental data. With reasonable assumptions the existence of an isobaric state of the nucleon is shown to be consistent also with a weak coupling theory. The phase shift for the s wave is of the order of magnitude of the experimental one, although the dependence on the energy is rather different.

The author is greatly indebted to Prof. G. WATAGHIN and M. VERDE for many stimulating discussions. He is also grateful to the Consiglio Nazionale delle Ricerche for financial support.

Note added in proof.

Prof. BERNARDINI called my attention to two papers of G. F. CHEW and J. S. BLAIR and G. E. CHEW (unpublished) dealing with the same problem. The Tamm-Dancoff method was applied by these authors with a pseudovector coupling and an extended source. They find a good agreement with the Fermi p phase shifts, while the s wave is not dealt with. No resonance is found by these authors.

APPENDIX I

Separation of the isotopic spin variables.

We define the M_1, M_2 matrices as:

$$(19) \qquad \begin{cases} \langle ir|M_1|js\rangle = \sum_k \tau^s_{ik}\tau^r_{kj}\,, \\ \langle ir|M_2|js\rangle = \sum_k \tau^r_{ik}\tau^s_{kj}\,. \end{cases}$$

Equation (5) may be written in the form:

$$(20) \qquad (W - E_p - \omega_p)a(\boldsymbol{p}) = \frac{g^2}{8\pi}\int [K_1(\boldsymbol{p}, \boldsymbol{q})M_1 + K_2(\boldsymbol{p}, \boldsymbol{q})M_2]a(\boldsymbol{q})\,\mathrm{d}^3 q\,.$$

We introduce the three matrices ω_k of isotopic spin of the pion, as defined by NAMBU and JAMAGUCHI (9). Using some elementary properties of the τ matrices, we get:

$$(21) \qquad M_1 = 1 + (\boldsymbol{\omega}\cdot\boldsymbol{\tau})\,, \qquad M_2 = 1 - (\boldsymbol{\omega}\cdot\boldsymbol{\tau})\,.$$

The projection operators for the states corresponding to $T = 3/2$ and $T = 1/2$ are given by:

$$(22) \qquad P_{3/2} = \frac{2 + (\boldsymbol{\omega}\cdot\boldsymbol{\tau})}{3}\,, \qquad P_{1/2} = \frac{1 - (\boldsymbol{\omega}\cdot\boldsymbol{\tau})}{3}\,.$$

Then from (21) and (22) it follows:

$$(23) \qquad M_1 = -P_{1/2} + 2P_{3/2}\,, \qquad M_2 = 3P_{1/2}\,.$$

From (20) and (23), expanding $a(\boldsymbol{p})$ as a series of eigenfunctions of the total isotopic spin we obtain equations (7*a*) and (7*b*).

APPENDIX II

The Fredholm and Schmidt methods for integral equations.

Let us consider the integral equation:

$$(24) \qquad \varphi(p) = f(p) + \int K(k; p, q)\,\varphi(q)\,\mathrm{d}q\,.$$

Both the Schmidt and the Fredholm methods give the solution in the form:

$$(25) \qquad \varphi(p) = f(p) + \frac{1}{D(k)}\int \Delta(k; p, q) f(q)\,\mathrm{d}q\,.$$

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(9) Y. NAMBU and Y. YAMAGUCHI: *Progr. Theor. Phys.*, **6**, 1000 (1951).

In the Fredholm method $D(k)$ and $\Delta(k; p, q)$ are given by:

$$\Delta(p, q) = K(p, q) - \int \begin{vmatrix} K(p, q) & K(p, r) \\ K(r, q) & K(r, r) \end{vmatrix} \mathrm{d}r + \ldots \tag{26}$$

$$D(k) = 1 - \int K(p, p)\,\mathrm{d}p + \iint \begin{vmatrix} K(p, p) & K(p, q) \\ K(q, p) & K(q, q) \end{vmatrix} \mathrm{d}p\,\mathrm{d}q - \ldots . \tag{27}$$

The Schmidt method consists in writing the kernel $K(k; p, q)$ in the form:

$$K(p, q) = \sum_i M_i(p) N_i(q) . \tag{28}$$

From (24) and (28) we obtain:

$$\varphi(p) = f(p) + \sum_i C_i M_i(p) , \tag{29}$$

where:

$$C_i = \int N_i(q) \varphi(q)\,\mathrm{d}q . \tag{30}$$

From (29) and (30) it follows that the constants C_i must satisfy the equations:

$$C_i = f_i + \sum_j K_{ij} C_j , \tag{31}$$

where:

$$\begin{cases} f_i = \int f(p) N_i(p)\,\mathrm{d}p , \\ K_{ij} = \int N_i(p) M_j(p)\,\mathrm{d}p . \end{cases} \tag{32}$$

Therefore $D(k)$ is equal to the determinant of the K_{ij} and $\Delta(k; p, q)$ is given by:

$$\Delta(p, q) = \sum_{ij} H_{ij} M_i(p) N_j(q) , \tag{33}$$

where H_{ij} is the minor of K_{ij}.

RIASSUNTO

Si applica il metodo non adiabatico di Tamm e Dancoff allo studio dell'urto nucleone-pione. Si ottiene un'equazione integrale che, risolta con metodi approssimati, fornisce risultati numerici in buon accordo con quelli sperimentali. Si prevede inoltre l'esistenza di uno stato isobarico del nucleone con spin 3/2 e spin isotopico 3/2.

Fluorescent Emission of Resonance Lines at Higher Pressures.

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(ricevuto il 19 Febbraio 1953)

Summary. — The aim of this paper is to show that some phenomena, whose origin has been hitherto ascribed to very loosely bound molecules, can be explained in a natural way on the assumption that their origin is due to the atoms. The phenomena in question were observed by Pringsheim and Jabłoński in saturated Na-vapour and by Pringsheim and Saltmarsh in saturated Hg-vapour at pressures beyond the point at which the resonance radiation is contracted as surface resonance at the primary light entrance wall of the container. The most striking of these phenomena are: the reappearance of the luminous beam (beam fluorescence) at higher pressures of the saturated vapour with a spectrum apparently consisting of normal resonance lines, the reappearance of polarisation of this fluorescence in spite of the fact that the « normal » resonance radiation is thoroughly depolarised at much lower pressures, the insensivity of this polarisation to the magnetic field, etc.

If the pressure of the saturated sodium vapour, in which the resonance radiation is excited, grows gradually, the character of the emission undergoes changes. At very low pressures ($p \sim 10^{-6}$ mm Hg) a well defined luminous beam appears along the path of the primary light (beam resonance). With increasing density the beam grows diffuser until at last the whole volume of the container is filled with luminosity (volume resonance). At pressures $\sim 10^{-4}$ mm Hg the luminous beam contracts to the entrance wall of the container (surface resonance). At still higher pressures ($\sim 10^{-3}$ mm Hg) the « beam fluorescence » reappers, provided the emission be excited by broadened resonance lines, or, even, by parts of a continuous spectrum adjacent to the

D-lines. The spectrum of this fluorescence seems to consist simply of the two D-lines (broadened and, possibly, shifted). In order to be distinguished from the « normal » resonance radiation, observed at lower pressures, a special name, « D-fluorescence », was given to this emission.

The polarisation of D-fluorescence is quite similar to that of D-resonance. If the D-fluorescence is excited by plane polarised light, D_1-line appears to be unpolarised, D_2-line partially polarised, the rate of its polarisation reaching 30 per cent at $p \approx 3.3 \cdot 10^{-4}$ mm Hg, although the depolarisation of the normal resonance is almost complete at $p \approx 10^{-4}$ mm Hg (1).

The magnetic field ~ 80 Oersted, no matter whether parallel or perpendicular to the direction of observation, does not affect appreciably the polarisation of D-fluorescence, while a field of 20 Oersted in the direction of observation does depolarise completely the normal resonance radiation (2).

Thus the behaviour of the D-fluorescence differs considerably from that of the D-resonance.

Only with respect to quenching by foreign gases, e.g. N_2, is the behaviour of D-fluorescence quite similar to that of D-resonance, the quenching cross section being much the same in both cases and larger than that for quenching the true molecular fluorescence of Na_2-molecules (1).

Phenomena, very similar to those observed in Na-vapour, were observed by PRINGSHEIM and SALTMARSH (3) in saturated Hg-vapour, although at somewhat higher pressures ($p \approx 10^{-3} \div 10^{-1}$ mm Hg) as in the first case. This difference of pressures is certainly due to the difference of transition probabilities of the corresponding resonance lines, the transition probability of the Hg intercombination resonance line (2357 Å) being some ten times smaller than that of the D-lines of Na.

Several arguments were given in papers published hitherto in favour of the view that the D-fluorescence as well as the analogous Hg-fluorescence are due to the very loosely bound molecules of the van der Waals type (v.d.W.-molecules) (4).

Three different hypotheses were discussed. First, the observed emission is a transition to the Rayleigh scattering of light of frequencies adjacent to that of the resonance line. Second, the emission is a normal resonance radiation with a distribution altered by Doppler effect and by pressure of the vapour. Third, the emission is due to the v.d.W.-molecules.

The first hypothesis has been rejected because of the fact that the emission is liable to quenching by admixture of a foreign gas, e.g. N_2, to the Na-vapour. Also the second hypothesis—the emission be a resonance radiation with

(1) A. JABŁOŃSKI and P. PRINGHSEIM: *Zeits. f. Phys.*, **70**, 593 (1931).
(2) A. JABŁOŃSKI and P. PRINGSHEIM: *Zeits. f. Phys.*, **73**, 281 (1931).

broadened and possibly shifted lines—seemed to be untenable. The argument was: the intensity distribution in a broadened and possibly shifted line observed at right angles to the direction of the primary light does not depend on the absorbed frequency. It seemed therefore quite impossible to understand how could such a radiation in conditions of experiments concerned reach the walls of the container without being reabsorbed several times. Besides, the insensibility of polarisation of D-fluorescence to the collisions and to the magnetic field seemed to be at variance with the hypothesis of atomic origin of the emission. The third hypothesis seemed to be the best match for the observed facts. The weakness of reabsorption of the emitted light could be reasonably explained by the smallness of concentration of the v.d.W-molecules emitting frequencies slightly out of tune with those absorbed by the resonance lines. The insensibility of polarisation of D-fluorescence to the collisions and to the magnetic field seemed also to corroborate the hypothesis of molecular origin of D-fluorescence, since, as well known, the molecular fluorescence is less liable to depolarisation by the above factors than the atomic one. From this point of view, the « reappearance » of polarisation is simply the « appearance » of polarised molecular fluorescence.

The « D-fluorescence bands » seemed to be analogous to the satellites of higher members of the principal series, observed by KUHN [5] in absorption spectra of alkali vapours.

There are, however, some features of the emission in question, which can be reconciled with the assumption of its molecular origin only with extreme difficulty.

The width of « D-fluorescence bands » does not exceed 0.5 Å. In order to explain within the frame of molecular hypothesis the narrowness of these bands one has to admit the identity of potential curves of the molecule in question in both electronic states responsible for the emission. In this case only a sequence $v' = v''$ would appear in the spectrum and the observed « band » could be very narrow indeed. It seems, however, that the rotational energy alone should cause the bands to have a larger width than that actually observed in D-fluorescence. Besides, it seems to be a rather unprobable accident that the potential curves belonging to three different electronic states of Na v.d.W.-molecules be practically identical, and, moreover, that the same occurs in the case of Hg v.d.W.-molecules. In the last case the situation is even more complicated since there exists in mercury vapour spectrum a band

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[3] P. PRINGSHEIM and O. SALTMARSH: *Proc. Roy. Soc.*, A **154**, 90 (1935).

[4] L. c. at [1], [2] and [3]. Cf. also the book by P. PRINGSHEIM: *Fluorescence and Phosphorescence* (New York, 1949), p. 238.

[5] H. KUHN: *Zeits. f. Phys.*, **76**, 782 (1932).

belonging to Hg_2 v.d.W-molecules and related to the resonance line 2537 Å This band (2 540 Å) is certainly not responsible for the fluorescence in question. Thus one should admit the existence of two kinds of v.d.W.-molecules in Hg.

A further difficulty is risen by the fact that the « D_1-band » appears to be unpolarised, whereas the molecular fluorescence should be partially polarised in conditions of performed experiments.

An attempt to determine the heat of dissociation of these hypothetical molecules has failed ([2]).

Clearly, the assumption of molecular origin of D-fluorescence (and of the similar fluorescence of Hg) does not lead to a wholly satisfactory explanation of all observed facts.

We now proceed to show in a qualitative way that this assumption is unnecessary indeed, since all experimental facts can be explained on the assumption of the atomic origin of the emission ([6]).

Let us discuss first the reappearance of the beam fluorescence. Let the absorption coefficient of a resonance line as a function of angular frequency ω be given by

$$\mu(\omega) \sim N \frac{\gamma + cN}{(\omega_0 - \omega)^2 + \left(\frac{\gamma + cN}{2}\right)^2}, \tag{1}$$

where γ is the natural half-width of the line, N the concentration of the absorbing atoms. It is assumed here that the shape of the line resulting from the combination of the natural intensity distribution and of the distribution caused by the pressure effects is given by the dispersion formula (1) ([7]). We neglect provisionally, for the sake of simplicity, the influence of the Doppler broadening and that of the increase of number of collisions with growing temperature — we assume the « pressure half-width » to be simply equal to cN, with $c = \text{const}$. As shall be shown below the conclusions of our considerations can be but strenghtened if these neglected effects are taken into account.

The formula (1) shows that for $cN \ll \gamma$ the absorption coefficient $\mu(\omega)$ is practically proportional to N throughout the whole spectral region of the line. It results therefrom a gradual contraction of the luminous beam with growing pressure of the saturated vapour. At sufficiently high pressures

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([6]) The present writer is very sorry indeed that he had not the opportunity to discuss the whole matter with Professor P. PRINGSHEIM.

([7]) The true shape of the line may differ considerably from that given by (1). However, for the rather qualitative discussion to be given here, the familiar dispersion formula may do sufficiently well.

the absorption coefficient of the core of the line becomes so large as to cause the resonance to concentrate at the entrance wall of the container (surface resonance). If the exciting line is not too narrow, a weak beam resonance, in addition to the surface resonance, appears ([8]) excited by absorption in the wings of the line). The beam resonance is weak as long as the absorption line is sharp, i.e. as long as $cN \ll \gamma$. This being so, the total energy absorbed by the wings of the line in the interior of the container is small. If cN becomes comparable with γ, the situation alters. The absorption line beginns to broaden. The spectral region of values of μ suitable for giving rise to the beam resonance grows broader. Therefore the energy absorbed from the primary light in the interior of the container becomes larger and the intensity of the luminous beam grows in spite of the losses of intensity caused by the reabsorption of frequences belonging to the core of the emitted line (only the wings of emitted line can reach the walls of the container without being reabsorbed several times).

For $cN \gg \gamma$ the absorption coefficient of the centre of line ($\omega = \omega_0$) ceases to increase with increasing concentration N of the saturated vapour:

$$\mu(\omega_0) \sim N \frac{\gamma + cN}{\left(\frac{\gamma + cN}{2}\right)^2} \approx \text{const.} \tag{2}$$

However, the absorption coefficient in the wings of the line, for $|\omega_0 - \omega| \gg \gg \gamma + cN$, begins to grow proportional to N^2.

The Doppler effect causes a further broadening of the line thus giving rise to an additional diminution of the absorption coefficient $\mu(\omega_0)$ at the centre of the line with increasing pressure of the saturated vapour, caused by the growth of the temperature ([9]). The, so far neglected, effect of temperature on pressure broadening, although probably not very significant, must cause a still further diminution of of $\mu(\omega_0)$, which, according to (1), should be independent of the temperature of the saturated vapour for $cN \gg \gamma$, does actually decrease, if the temperature and thus also the density of the saturated vapour is risen. This result seems to be paradoxical. That is probably the cause, why no attempt has been made hitherto to explain the reappearance of the beam fluorescence by pressure effects.

The strange dependence of the polarisation of resonance radiation on the pressure of the saturated vapour and its insensibility to the influence of

([8]) R. J. Strutt: *Proc. Roy. Soc.*, A **96**, 272 (1919).

([9]) The Doppler effect alone should cause the $\mu(\omega_0)$ to be inversely proportional to the square root of the absolute temperature.

magnetic field can be explained on ground of the theory of resonance fluorescence, given by WEISSKOPF ([10]).

We assume that the enormous cross section for depolarisation of the « true » resonance radiation by the own pressure of the vapour is due to the « coupling » between the atoms of the same kind. According to WEISSKOPF this coupling takes place when the mutual distances of neighbouring atoms become comparable with the wave length of the resonance line. This coupling constitutes an anologue to the classical coupling of the radiating oscillators situated near each other. At greater densities of the vapour, when several atoms are contained within a cube with edges equal to the resonance wavelength, the coupling forces cease to affect the resonance radiation. The mutual perturbations of the atoms cancel each other. The natural conclusion driven out of this Weisskopf's picture is that at higher pressures the depolarisation cross section must become smaller and approach to that for depolarisation by foreign molecules in which case no coupling forces can exist.

Hence the qualitative picture may be given as follows. At pressures at which the mutual distances of the atoms are large compared with the wavelength of resonance radiation there is no depolarisation by collisions. At pressures at which the interatomic distances become comparable with the wave length of the resonance radiation the coupling forces cause a strong depolarisation of radiation. At still higher pressures when the coupling forces begin to cancel each other the polarisation reappears and after reaching a maximum at a certain pressure begins to decrease slowly. This decrease is due to the depolarising collisions with a cross section comparable with that for the collisions with foreign atoms or molecules.

The insensibility of polarisation to the influence of a magnetic field directed along the exciting beam can be explained as due to the broadening of the Zeeman levels. If the Zeeman shifts are small compared with widths of the levels, no change of probability of transfer of atoms from one Zeeman level to another by collisions is to be expected.

However, the most interesting is the case in which the magnetic field is parallel to the direction of observation. In this case the plane of polarisation of the « true » resonance radiation is rotated, and, besides, the rate of polarisation reduced ([11]). This phenomenon was explained by HANLE on the assumption that the Larmor precession is carried out in the magnetic field by a linear oscillator, whose amplitude decreases due to the radiation damping. This picture can give no answer on the question why in the case of the D-fluorescence no such effects occur. Here, however, Weisskopf's theory ([11])

([10]) V. WEISSKOPF: *Ann. der Phys.*, **9**, 23 (1931).
([11]) W. HANLE: *Zeits. f. Phys.*, **30**, 93 (1924).

gives an explanation. The rotation of the plane of polarisation (which is also the cause of the depolarisation) is not the same throughout the whole width of the line, as in the semi-classical theory of Hanle. It depends on frequency. The angle of rotation of the plane of polarisation ϑ as a function of frequency $\nu = 2\pi/\omega$ is given by

$$\text{tg } 2\vartheta = \frac{2\gamma\Delta}{(\nu_0 - \nu)^2 + \gamma^2 - \Delta^2}, \tag{3}$$

where $\nu_0 = \omega_0/2\pi$ is the frequency of the centre of the resonance line, γ the half-width of the upper electronic level (i.e. the transition probability from the excited to the ground level) and $\Delta = (e/m)(H/4\pi c)$ the Larmor frequency in a magnetic field of H oersted (e and m are charge and mass of electron, c the velocity of light). The formula (3) is derived for resonance lines with natural intensity distribution, i.e. lines not broadened by pressure and Doppler effect. Thus, if applied to our case, it can lead merely to qualitative conclusions.

At very low pressures, at which there is no appreciable reabsorption of the line, the mean value of ϑ (over all frequencies) is observed. There results from (3) the mean value $\bar{\vartheta}$ given by

$$\text{tg } 2\bar{\vartheta} = \frac{\Delta}{\gamma}. \tag{4}$$

It is however clear that if only the wings of the line reach the walls of the container, the observed rotation must be smaller than that given by (4). Inspection of (3) shows that the rotation is practically nil, if the only frequencies observed are those sufficiently different from ν_0.

Since the depolarisation of the resonance radiation by magnetic field is directly related to the rotation of the plane of polarisation, no appreciable depolarisation of D-fluorescence by magnetic field is to be expected.

The fact that the D-fluorescence is not appreciably depolarised by magnetic field may be considered as an experimental confirmation of one of the conclusions of the theory of Weisskopf.

It seems that in the light of the above considerations it can be stated that the D-fluorescence of the sodium vapour and similar phenomena observed in mercury vapour are due to atoms rather than to molecules.

RIASSUNTO (*)

Scopo del presente lavoro è di dimostrare che alcuni fenomeni, la cui origine è stata finora attribuita a molecole legate molto debolmente, possono essere interpretate in modo del tutto naturale attribuendone l'origine agli atomi. I fenomeni in questione furono osservati da PRINGSHEIM e JABŁOŃSKI in vapore di Na saturo e da PRINGSHEIM e SALTMARSH in vapore di Hg saturo a pressioni eccedenti il punto in cui la risonanza di radiazione si contrae alla parete del recipiente su cui incide la radiazione primaria in forma di radiazione superficiale. I più appariscenti di tali fenomeni sono: il riapparire del fascio luminoso (fluorescenza a fascio) a pressioni superiori del vapore saturo con uno spettro consistente apparentemente di linee di risonanza normali; il riapparire della polarizzazione di tale fluorescenza ad onta del fatto che la radiazione di risonanza « normale » si depolarizza completamente a pressioni assai inferiori; l'insensibilità di questa polarizzazione al campo magnetico, ecc.

(*) *Traduzione a cura della Redazione.*

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Non-Linearities in the Strong-Focusing Accelerator (*).

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(ricevuto il 14 Marzo 1953)

Summary. — The effects of the non-linearity of the magnetic field, due to the fact that it must satisfy the Maxwell equations, are estimated with particular regard to a strong-focusing accelerator. They are shown to be negligible if compared with other causes of error, so that the customary linear approximation can be safely used to discuss focusing properties. The Appendix contains formulae which exhibit the symmetry properties of the magnetic field and can be used for its calculation if the pole shapes are known, or conversely.

1. – It is customary to study the focusing properties of the magnetic field in an accelerator by means of equations linearized in the vertical and radial displacement from the stable orbit, the neglected terms being regarded as small perturbations. This procedure is not quite strict mathematically, as it is well known that conditions thus derived, in general, are necessary but not sufficient to secure stability. The novelty of design of the strong-focusing accelerator proposed by Courant *et al.* [1] requires that careful consideration be given to all factors entering into play, in particular to all perturbing causes which may prevent the particles from staying reasonably close to the stable orbit. It has seemed therefore convenient to estimate quantitatively the

(*) This work is a revised and abridged version of a CERN report (CERN/T/ERC/2). The suppressed part is to be substituted with the accompanying work by A. Turrin. and the author.

(1) E. Courant, S. Livingston and H. Snyder: *Phys. Rev.*, **88**, 1190 (1952).

effects of the hitherto neglected non-linearities. The non-linearities to be feared are of two essentially different types: those deriving from imperfect construction of the machine and those deriving from the fact that a linear field cannot be actually realized, as it would not satisfy the Maxwell equations. The study of the first is essentially a technical design problem and will not be considered here.

We estimate here in the customary adiabatic approximation the effects of the second, by taking a field which reduces to the customary linear approximation but for non-linear terms necessary for it to satisfy Maxwell's equations.

One further remark is that we are interested only in stability (i.e. boundedness of the orbits within the chamber), and not in periodicity of the orbits. We obtain first the conditions for stability in the standard case (machine with only one circular sector, focusing both radially and vertically), then proceed to estimate the effects of non-linearity in the strong-focusing accelerator by introducing, to speed up calculations, some approximations which do not affect appreciably the final numerical conclusions. We treat in detail only the case of vertical motion in a focusing sector, which suffices to show us what to expect in general.

Numerical computations are made by taking a field $B = 10\,000$ gauss, with a gradient $|n| = 5\,000$ and a radius of the stable orbit $R = 10\,000$ cm.

The conclusion we reach is that the deviations due to non linearity of the actual orbits from the orbits resulting from the linearized equations are certainly smaller than those due to imperfect construction of the machine, so that the treatment of the problem with the standard perturbation procedure is well justified.

2·1. – The equations of motion, for a given circular sector, derive from the Hamiltonian:

$$E = c\sqrt{m_0^2c^2 + P_r^2 + P_z^2 + \left(\frac{P_\varphi}{r} - \frac{e}{c}A\right)^2}\,, \tag{1}$$

where cylindrical coordinates are used (the stable orbit is in the plane $z = 0$; r is the distance from the center of the sector) and $A = A_\varphi$ is the only component $\neq 0$ of the vector potential. In (1)

$$P_r = m\dot{r} = m_0/\sqrt{1 - v^2/c^2}\,\frac{\mathrm{d}r}{\mathrm{d}t}\,, \tag{2}$$

$$P_z = m\dot{z}\,, \tag{2'}$$

$$P_\varphi = mr^2\dot{\varphi} + \frac{e}{c}rA\,, \tag{2''}$$

so that the equations are (see end of this section):

$$(3)\qquad \begin{cases} \dot{P}_r = mr\dot{\varphi}^2 - \dfrac{e}{c} r\dot{\varphi} B_z \,, \\ \dot{P}_z = \dfrac{e}{c} r\dot{\varphi} B_r \,, \\ P_\varphi = \text{const} \,. \end{cases}$$

With our hypotheses, $\partial A/\partial q = 0$ and $\Delta \times \boldsymbol{B} = 0$ in each sector. We have thus two integrals of motion, (1) and (2''). Call R the radius of the stable orbit, and take as new coordinate

$$s = Rq \,.$$

Denoting with a prime the derivative with respect to s, (1) becomes

$$(4)\qquad \frac{E^2 - m_0^2 c^4}{c^2} = P^2 = \left[\frac{P_\varphi}{r} - \frac{e}{c} A\right]^2 \cdot \left[1 + \frac{R^2}{r^2}(r'^2 + z'^2)\right] .$$

Since $B_r(r, 0) = 0$ (see App.), one can set $z = 0$ in the preceding equations to obtain the equation of the radial motion. Denoting with r_0, r'_0 the values of r, r' at $t = t_0$, the relations

$$\frac{P_\varphi}{r_0} = mr_0\dot{\varphi}_0 + \frac{e}{c} A(r_0, 0) \,,$$

$$P^2 = m^2 r_0^2 \dot{\varphi}_0^2 \left(1 + \frac{R^2 r'^2_0}{r_0^2}\right) ,$$

can be used to eliminate P_φ from (4), to obtain

$$(5)\qquad r'^2 = \frac{r^2}{R^2}\left[\frac{1}{\left(\dfrac{r_0^2}{\sqrt{r_0^2 r^2 + R^2 r_0'^2 r^2}} + \dfrac{e}{Pc}\left\{A(r_0, 0)\dfrac{r_0}{r} - A(r, 0)\right\}\right)^2} - 1\right] ,$$

which is the desired equation. Stability can be investigated from (5), with the purely algebraic criteria we shall apply to eq. (6) below, and no more than that is required in the case of a machine of standard type, consisting of only one focusing circular piece. The direct study of eq. (5) proves however rather cumbersome. We prefer therefore to follow an approximate procedure, which does not alter appreciably our final results and is simplest for the case of vertical motion. That is, we «define» the vertical motion by setting

$r = R = \text{const.}$ in (1) and (2″) and proceed as before to get the equation

$$z'^2 = \frac{1}{\left[\cos\alpha_0 + \frac{e}{Pc}\left\{A(R, z_0) - A(R, z)\right\}\right]^2} - 1 , \tag{6}$$

where $\operatorname{tg}\alpha_0 = z_0'$ [2].

Before specializing (6) to the case of interest, it is well to remark explicitly that (6) itself can be put to use to investigate whether a given field A has satisfactory focusing properties; conversely, it offers a simple algebraic criterion to look for fields of this sort. This can be seen quite simply by observing that a field, in a focusing sector (fs) must be capable of bending trajectories (or of a suitable bundle of them) towards the plane $z = 0$. There must be therefore in a fs points in which $z' = 0$: they are the solutions of the equation in z:

$$\cos\alpha_0 + \frac{e}{Pc}\{A(R, z_0) - A(R, z)\} = \pm 1 . \tag{7}$$

If A is given, one can solve (7) and test whether its solutions $\bar{z}$ satisfy a condition $\bar{z}^2 < D^2$, D being half the vertical aperture. The smaller the D required for this to happen, the better the focusing properties of the field. This procedure is studied in detail in the case of interest to us.

The particles (positively charged) are supposed to move counterclockwise; the proper deflection is then assured ($\boldsymbol{F} = (e/c)\boldsymbol{v}\times\boldsymbol{B}$) if $\boldsymbol{B}$ $(R, 0)$ points downward. To have all quantities e, $\dot{\varphi}$, B_0 positive, we take a left-handed frame of reference with the z-axis pointing downward and the r-axis pointing outwards. In this frame we have

$$B_z = -\frac{1}{r}\frac{\partial}{\partial r}(rA) , \qquad B_r = \frac{\partial A}{\partial z} .$$

We take A as given by

$$A = -\frac{1}{2}B_0\left(1 + \frac{n}{2}\right)r - \frac{B_0 n}{2R^2}\left(rz^2 - \frac{1}{4}r^3\right) . \tag{8}$$

(A here is $-A$ of App.).

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[2] The error thus introduced (which corresponds to the addition of a small radial constraint to eq. (3)) could be eliminated by writing down some more inequalities of the type to be used later, but we forego doing so as nothing would be practically changed in our conclusions.

2·2. – The focusing properties of the field defined by (8) can be studied from eq. (7). This becomes

$$\cos\alpha_0 + \frac{eB_0 n}{2cPR}[z^2 - z_0^2] = \pm 1 . \tag{9}$$

We distinguish two cases: *a*) $n > 0$; *b*) $n < 0$. The parameter which plays a fundamental role in the sequel is

$$\beta = \frac{eB_0|n|}{2cPR} ;$$

β is a constant of the machine, which can be estimated as follows. Take, in our case,

$$B_0 = 10^4 \text{ gauss} ; \qquad |n| = 5\cdot 10^3 , \qquad R = 10^4 \text{ cm} .$$

P is of the order of magnitude of P_0 on the stable orbit. For the latter we have, from the energy integral (5) and from (3) ($\dot{P}_r \approx 0$):

$$P_0 = mR\dot{\varphi}_0 = \frac{e}{c} RB_z(R, 0) = \frac{e}{c} RB_0 ;$$

it follows:

$$\beta \approx \frac{eB_0|n|}{2cP_0R} = \frac{|n|}{2R^2} = 2{,}5\cdot 10^{-5}\text{cm}^{-2} .$$

Differentiating (6) we obtain

$$z'' = -\frac{2}{\left[\cos\alpha_0 + \dfrac{n}{|n|}\beta(z^2 - z_0^2)\right]^3} \cdot \frac{n}{|n|}\beta z , \tag{10}$$

which tells whether we are dealing with maxima or minima.

[sf] *a*) The solutions corresponding to the + sign in eq. (9) are:

$$\bar{z} = \pm\sqrt{\frac{1-\cos\alpha_0}{\beta} + z_0^2} , \tag{11}$$

they can satisfy the condition

$$\bar{z}^2 < D^2 . \tag{12}$$

One gets a maximum or a minimum according to the sign $\pm$ given to (11).

The solution corresponding to the — sign in eq. (9) is imaginary in our range of values and we do not have to consider it.

[dfs] *b*) Consider again a condition of type (12) written now as

$$z^2 < d^2 \leqslant D^2 \,. \tag{12'}$$

It is compatible only with the solution

$$\bar{z} = \pm \sqrt{-\frac{1-\cos\alpha_0}{\beta} + z_0^2} \qquad (+ \text{ sign in (9)}) \,. \tag{13}$$

We find then, however, the additional limitation (reality condition for $\bar{z}$):

$$\cos\alpha_0 + \beta z_0^2 \geqslant 1 \,. \tag{14}$$

For $z_0 = 0$ (except for the stable orbit $\alpha_0 = 0$) (14) is impossible; for $|z_0| > 0$ it can be satisfied, and one sees then easily from (10) and from $\bar{z}^2 \leqslant z_0^2$ that:

if $z_0 > 0$, $z_0' \leqslant 0$ and (13) corresponds to a minimum $\geqslant 0$,

if $z_0 < 0$, $z_0' \geqslant 0$ and (13) corresponds to a maximum $\leqslant 0$.

Therefore: for $n < 0$ either we have that the condition (12) cannot be satisfied at all, or we find that it can only be satisfied by orbits which ultimately diverge from the stable orbit: we are in a defocusing sector (dfs).

The angular spread is determined easily at all points from eq. (6), which now reads

(fs): $$\cos\alpha = \cos\alpha_0 + \beta(z^2 - z_0^2) \,,$$

(dfs): $$\cos\alpha = \cos\alpha_0 - \beta(z^2 - z_0^2) \,,$$

having put $\cos\alpha = 1/\sqrt{1 + z'^2}$.

3·1. – In studying what happens in a fs of length l a quantity of interest is that defined by relation (11); we term it z_m and take:

$$z_m = \pm \sqrt{z_0^2 + \frac{2}{\beta}\sin^2\frac{\alpha_0}{2}} \quad \text{(max. of a trajectory starting from } z_0,\ \alpha_0\text{)}.$$

It is sufficient to consider condition (12) for z_m only:

$$z_0^2 + \frac{2}{\beta}\sin^2\frac{\alpha_0}{2} < D^2 . \tag{15}$$

(15) guarantees that the particles enter the fs so as to remain always in it. As far as a fs is concerned, (15) is all we need to secure that the particles will not hit the chamber walls.

With $b = \cos\alpha_0 - \beta z_0^2$, the integrals of (6) are of type:

$$s = \pm\int \frac{b + \beta z^2}{\sqrt{1-(b+\beta z^2)^2}}\,\mathrm{d}z + \text{const.}$$

(the sign is to be fixed by continuity arguments). We must consider separately two cases: *a*) trajectories which always increase (decrease); *b*) trajectories which bend *once*, after reaching $\pm z_m$ (we shall see that *a*) and *b*) exhaust all possibilities in the case we consider). Subcases are also to be considered, according to the initial angle, as is shown below.

a_1) $z_0' > 0$ $(z_1 > z_0)$ (Includes also case $z_0' = 0$, $z_0 = -z_m$):

$$l = \int_{z_0}^{z_1} \frac{b + \beta z^2}{1-(b+\beta z^2)^2}\,\mathrm{d}z .$$

Fundamental to the following is the substitution

$$z = z_m\cos\psi = \sqrt{\frac{1-b}{\beta}}\cos\psi , \tag{16}$$

which, with

$$k = \sqrt{\frac{1-b}{2}}$$

yields

$$l = \frac{1}{\sqrt{2\beta}}\int_{\arccos(z_1/z_m)}^{\arccos(z_0/z_m)} \frac{1-2k^2\sin^2\psi}{\sqrt{1-k^2\sin^2\psi}}\,\mathrm{d}\psi .$$

a_2): again from 16) one gets $(z_0' < 0;\ z_1 < z_0$. Also valid for $z_0' = 0$, $z_0 = +z_m$):

$$l = \frac{1}{\sqrt{2\beta}}\int_{\arccos(z_0/z_m)}^{\arccos(z_1/z_m)} \frac{1-2k^2\sin^2\psi}{\sqrt{1-k^2\sin^2\psi}}\,\mathrm{d}\psi ;$$

$b_1)$ $(z_0' > 0$ Also: $z_0' = 0,\ z_0 = -z_m)$:

$$l = \frac{1}{\sqrt{2\beta}}\left[\int_0^{\arccos(z_0/z_m)} + \int_0^{\arccos(z_1/z_m)} \frac{1-2k^2\sin^2\psi}{\sqrt{1-k^2\sin^2\psi}}\,d\psi\right],$$

$b_2)$: $(z_0' < 0.$ Also: $z_0' = 0,\ z_0 = z_m)$:

$$l = \frac{1}{\sqrt{2\beta}}\left[\int_{\arccos(z_0/z_m)}^{\pi} + \int_{\arccos(z_1/z_m)}^{\pi} \frac{1-2k^2\sin^2\psi}{\sqrt{1-k^2\sin^2\psi}}\,d\psi\right].$$

We use now the inequality

$$(17)\qquad (1-2k^2)(\varphi_2-\varphi_1) \leqslant \int_{\varphi_1}^{\varphi_2} \frac{1-2k^2\sin^2\psi}{\sqrt{1-k^2\sin^2\psi}}\,d\psi \leqslant \frac{\varphi_2-\varphi_1}{\sqrt{1-k^2}}, \qquad (\varphi_2 \geqslant \varphi_1).$$

(17), applied to the formulae $a_1) \ldots b_2)$, yields:
$a_1)$:

$$(1-2k^2)\left|\arccos\frac{z_0}{z_m} - \arccos\frac{z_1}{z_m}\right| \leqslant \sqrt{2\beta}\,l \leqslant \frac{1}{\sqrt{1-k^2}}\left|\arccos\frac{z_0}{z_m} - \arccos\frac{z_1}{z_m}\right|,$$

or:

$$(18)\qquad z_m\sin(\sqrt{1-k^2}\sqrt{2\beta}\,l+\varepsilon) \leqslant z_1 \leqslant z_m\sin\left(\frac{1}{1-2k^2}\sqrt{2\beta}l+\varepsilon\right),$$

with:

$$\varepsilon = \arcsin\frac{z_0}{z_m};$$

$a_2)$:

$$(19)\qquad -z_m\sin\left(\frac{1}{1-2k^2}\sqrt{2\beta}\,l-\varepsilon\right) \leqslant z_1 \leqslant -z_m\sin(\sqrt{1-k^2}\sqrt{2\beta}l-\varepsilon);$$

$b_1)$:

$$(20)\qquad z_m\sin\left(\frac{1}{1-2k^2}\sqrt{2\beta}l+\varepsilon\right) \leqslant z_1 \leqslant z_m\sin(\sqrt{1-k^2}\sqrt{2\beta}l+\varepsilon);$$

$b_2)$:

$$(21)\qquad -z_m\sin(\sqrt{1-k^2}\sqrt{2\beta}l-\varepsilon) \leqslant z_1 \leqslant -z_m\sin\left(\frac{1}{1-2k^2}\sqrt{2\beta}l-\varepsilon\right).$$

3·2. – The change of the argument of the sines in passing from formulae (18) to (20) and (19) to (21) can be understood quite simply:

Case $z_0' > 0$: $a_1)$, $b_1)$.

Consider the length $\bar{l}$ corresponding, for given z_0, α_0, to the excursion from z_0 to z_m. For it we find:

$$\sqrt{2\beta}\,\bar{l} = \int_0^{\arccos(z_0/z_m)} \frac{1-2k^2\sin^2\psi}{\sqrt{1-k^2\sin^2\psi}}\,\mathrm{d}\psi\,,$$

$$\cos\frac{\sqrt{2\beta}\,\bar{l}}{1-2k^2} \leqslant \frac{z_0}{z_m} \leqslant \cos\sqrt{1-k^2}\sqrt{2\beta}\,\bar{l}\,,$$

$$\frac{\sqrt{2\beta}\,\bar{l}}{1-2k^2} \geqslant \frac{\pi}{2} - \varepsilon \geqslant \sqrt{1-2k^2}\,\sqrt{2\beta}\,\bar{l}\,, \quad \text{or} \quad \frac{\pi}{2} - \varepsilon \approx \sqrt{2\beta}\,\bar{l}\,,$$

which gives

$$\varepsilon + \sqrt{2\beta}\,l > \frac{\pi}{2} \qquad \text{for } l > \bar{l}\,,$$

and explains the transition from (18) to (20).

Case $z_0' < 0$: $a_2)$, $b_2)$.

We find here, from $\sqrt{2\beta}\,\bar{l} = \int_{\arccos(z_0/z_m)}^{\pi} \frac{1-2k^2\sin^2\psi}{\sqrt{1-k^2\sin^2\psi}}\,\mathrm{d}\psi\,,$

$$\frac{\pi}{2} + \varepsilon \approx \sqrt{2\beta}\,\bar{l}\,,$$

which shows why (19) becomes (21), since, for $l > \bar{l}$, $\sqrt{2\beta}\,l - \varepsilon > \pi/2$.

For completeness, we add a few more considerations on the nature of the *exact* (to within our approximations) motion of a given particles in a fs. Take $z_0' > 0$:

consider the length

$$L = \int_{z_0}^{z_m} - \int_{z_m}^{-z_m} + \int_{-z_m}^{z_0} = \int_0^{\arccos(z_0/z_m)} + \int_0^{\pi} + \int_{\arccos(z_0/z_m)}^{\pi} = \frac{2}{\sqrt{2\beta}}\int_0^{\pi} \frac{1-2k^2\sin^2\psi}{\sqrt{1-k^2\sin^2\psi}}\,\mathrm{d}\psi =$$

$$= \frac{4}{\sqrt{2\beta}}[2E(k) - F(k)] = \frac{2\pi}{\sqrt{2\beta}}\left(1 - \frac{3k^2}{4} + \frac{15k^4}{64} \ldots\right) \approx \frac{2\pi}{\sqrt{2\beta}}\,.$$

In the formula just written, we have introduced the complete elliptic integrals of the first two kinds. After a path L, the particle can again repeat the same motion, if we consider for a moment a standard machine consisting only of one complete circular fs: the condition for periodicity of the vertical motion becomes (after n revolutions in the machine:

$$n2\pi R = \frac{4}{\sqrt{2\beta}}[2E(k) - F(k)] .$$

k, however, depends upon z_0 and α_0, so that, for given z_0 (or α_0) only a *discrete* set of orbits can be periodic in the large. All such orbits, however, regardless of periodicity, would be stable in our sense, i.e. always stay inside the vacuum chamber.

The length corresponding to a complete half period is

$$\frac{L}{2} = \frac{\pi}{\sqrt{2\beta}}\left(1 - \frac{3}{4}k^2 + ...\right) \approx \frac{\pi}{\sqrt{2\beta}} .$$

It depends negligibly on z_0 and α, due to the smallness of k: $L \approx 4.42$ m in our case. Since the machine is planned with $l < 4.42$ m, we are sure that cases *a*) and *b*) actually exhaust all possibilities.

3·3. – In conclusion, we can estimate the effects of non-linearity from formulae (18)-(21). The maximum displacement $\overline{\Delta z_m}$ of the actual orbit from the orbit corresponding to the linearized equation (obtained by setting $k = 0$ in (18)-(21); z_m remains practically unchanged) is certainly smaller, around the value $\bar{l}$ of l for which the maximum occurs ($\sqrt{2\beta}\,\bar{l} \pm \varepsilon = \pi/2$ (see 2). This is the less favorable case) than:

$$\overline{\Delta z_m} = z_m \left| \sin\left(\frac{1}{1 - 2k^2}\sqrt{2\beta}\,\bar{l} \pm \varepsilon\right) - 1 \right| \approx z_m \left| \sin\left(\frac{\pi}{2} + 2k^2\sqrt{2\beta}\,l\right) - 1 \right| \approx$$

$$\approx 4z_m\beta\,\bar{l}^2k^4 < 4D\beta l^2k^4 < \beta^3D^3l^2$$

$$\left(\text{since, from (15)}, \quad k^2 = \frac{1 - \cos\alpha_0 + \beta z_0^2}{2} < \frac{\beta D^2}{2}\right).$$

We obtain thus, with $D = 2 \div 3$ cm, $\overline{\Delta z_m} \approx 10^{-8}$ cm. This is certainly small enough to justify the conclusion that the effects of non-linearities of the type we have been considering are negligible in comparison with those arising from imperfect construction. Results of the same order of magnitude are also to

be expected for the other modes of motion, which can be studied in the same way.

The author is glad to acknowledge his indebtedness to Prof. N. Bohr for stimulating his interest in this argument, and to Dr. M. Sands for many clarifying discussions.

Appendix

Symmetry properties of the magnetic field.

Let r, z, φ be cylindrical coordinates. The stable (circular, in each section with field) orbit is at $r = R, z = 0$. $B_\varphi(r, z) = 0$. $\boldsymbol{B} = \boldsymbol{B}(r, z)$ has to satisfy the Maxwell eq.s:

$$(1) \qquad \begin{cases} \dfrac{\partial B_r}{\partial z} - \dfrac{\partial B_z}{\partial r} = 0 , \\ \dfrac{\partial B_r}{\partial r} + \dfrac{1}{r} B_r + \dfrac{\partial B_z}{\partial r} = 0 , \end{cases}$$

and the symmetry requirements:

$$(2) \qquad \begin{cases} B_z(r, z) = B_z(r, \quad z) , \\ B_r(r, z) = - B_r(r, -z) . \end{cases}$$

Let

$$B_r = \sum_{i,k=0}^{\infty} \lambda_{ik} r^i z^k ,$$

$$B_z = \sum_{i,k=0}^{\infty} \mu_{ik} r^i z^k .$$

Then, from (1) and (2):

$$\lambda_{i,2z} = \mu_{i,2z+1} = 0$$

and:

$$(3) \qquad \begin{cases} i(i+2)\lambda_{i+1,k} + (k+2)(k+1)\lambda_{i-1,k+2} = 0 , \\ (i+2)^2 \mu_{i+2,k-1} + (k+1)^2 \mu_{i,k+1} = 0 . \end{cases}$$

Denote now with $f(z)$ an arbitrary analytic *even* function of z, and with D the operator $\mathrm{d}/\mathrm{d}z$. The most general field compatible with (1), (2) and (3) is easily shown to be of the form

$$(4)\qquad \begin{cases} B_r(r,z) = -\left[\frac{1}{2} rD - \frac{r^3}{2^3\cdot 4} D^3 + \frac{r^5}{2^2\cdot 4^2\cdot 6} D^5 - \ldots\right] f(z) = -J_1(rD)f(z) \\ B_z(r,z) = \left[1 - \frac{r^2}{2^2} D^2 + \frac{r^4}{2^2\cdot 4^2} D^4 \ldots\right] f(z) = J_0(rD)f(z)\,, \end{cases}$$

and to derive from a potential vector $\boldsymbol{A} \equiv (0, 0, A_\varphi)$, where:

$$A_\varphi = A = \left(\frac{r}{2} - \frac{r^3}{2^2\cdot 4} D^2 + \ldots\right) f(z) = D^{-1}J_1(rD)f(z)\,.$$

A is determined to within an additive term const./r, which, however, can be neglected as it gives clearly no trouble in the integration of the equations of motion.

Write now:

$$r = R + x\,.$$

The field which is closest to the standard form

$$(5)\qquad \begin{cases} B_z = B_0\left(1 - n\,\frac{x}{R}\right), \\ B_r = -B_0 n\,\frac{z}{R}\,, \end{cases}$$

and is still compatible with Maxwell's eq.s is given by

$$(6)\qquad \begin{cases} B_z = B_0\left(1 - n\,\frac{x}{R} + \frac{n}{R^2}\left[z^2 - \frac{x^2}{2}\right]\right), \\ B_r = -B_0\left(n\,\frac{z}{R} + n\,\frac{zr}{R^2}\right), \end{cases}$$

which derives from the vector potential (using again r)

$$(7)\qquad A = \frac{1}{2} B_0\left(1 + \frac{n}{2}\right) r + B_0\,\frac{n}{2R^2}\left(rz^2 - \frac{1}{4} r^3\right).$$

(6) and (7) are yielded by taking in (4)

$$f(z) = a_0 + a_1 z^2$$

and matching the constants a_0 and a_1 to fit (5).

Formulae (4) are quite convenient to calculate the magnetic field corresponding to given pole shapes, as one sees immediately by taking, for instance,

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp[izu] g(u)\, du$$

with $g(u)$ even. They permit also to answer a question of some practical interest: they show that it is sufficient to measure with sufficient accuracy the component $B_z(r, z)$ in the plane $z = 0$ to determine the values of both components at all points r, z. This suffices, indeed, to determine the constants a_k which appear in the development of $f(z)$ in powers of z^k to the wanted order of approximation, and the knowledge of $f(z)$ gives, as we have seen, all the information which is needed.

RIASSUNTO

Vengono valutati gli effetti della non linearità del campo magnetico, dovuta al fatto che questo deve soddisfare le equazioni di Maxwell, con particolare riferimento ad un acceleratore a « focalizzazione forte ». Essi risultano trascurabili rispetto ad altre cause di errore, sì da giustificare l'uso della consueta approssimazione lineare nello studio delle proprietà focalizzanti di un campo. In Appendice sono date alcune formule che pongono in evidenza le proprietà di simmetria del campo magnetico e possono servire al calcolo di questo se la forma delle espansioni polari è nota, o viceversa.

Stability and Periodicity in the Strong-Focusing Accelerator (*).

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Summary. — A simple formalism is developed for the study of a strong-focusing accelerator of arbitrary geometry. It is shown to yield readily the necessary and sufficient conditions for stability and for periodicity (with any prescribed period) of the orbits. Criteria are also given which allow the determination of the range of initial conditions for which the orbits are stable.

1. – We consider an accelerator consisting of circular sectors which, for a given mode of motion, act as focusing (fs) or defocusing (dfs) and of straight field-free sectors (ffs) disposed according to some periodic pattern which we need not specify. The behaviour of an accelerator of thys type [1] has been studied till now only for some special patterns and with methods which do not easily allow an understanding of its general properties. In particular, the effects of ffs's have been considered as perturbations, while, as we shall show, it is possible to treat them, quite simply, on the same footing as the fs's and dfs's.

Our treatment is valid as far as it is legitimate to adopt, for the description of the particle motion, the customary linearized equations for the vertical and radial displacements from the stable orbit, in the adiabatic approximation. Arguments in support of this procedure are given in the preceding work [2],

(*) This work has been performed at the University of Rome.

(1) E. Courant, S. Livingston and H. Snyder: *Phys. Rev.*, **88**, 1190 (1952).

(2) E. R. Caianiello: *Nuovo Cimento*, **10**, 581 (1953).

where the effects of non-linearity of the field are estimated to be unimportant in comparison with those arising from imperfect construction of the machine.

We study two separate questions:

1) the conditions under which *stability* (i.e. boundedness within the chamber walls) is guarantied for all the orbits of a given bundle;

2) the conditions which must be fulfilled in order that the orbits be *periodic* after traversing an arbitrary number of sectors. We shall find here, as a particular case, the well known periodicity after eight sections.

Only 1) is important if we restrict ourselves to the case of an «ideal» machine, as we do throughout this work. Knowledge of 2) becomes, however, important when considering the actual case, in which one wants to estimate the tolerances of the machine, and needs know whether small systematic perturbations are enhanced by resonances or not. This question is treated in detail in a very interesting work by SANDS and TOUSCHEK (3).

We consider only the vertical motion; everything we say applies also to the case of radial motion, due to the large value of the field gradient (n). We take $|n|$ constant in all fs's and dfs's: see, however, Part. 3·2).

2·1. – If z denotes the vertical displacement from the stable orbit, R the radius of this orbit, $|n|$ the magnitude of the field gradient, the standard linearized equations for z are:

in a fs:

$$z'' + \frac{|n|}{R^2} z = 0 , \tag{1}$$

in a dfs:

$$z'' - \frac{|n|}{R^2} z = 0 . \tag{2}$$

The derivative is taken with respect to the arc length $l = R\varphi$. (Comparison of (1) and (2) with formula (10) of ref. (2), from which they can be immediately derived, gives an idea of the approximation which is made by neglecting non-linearities of the field: $2\beta \approx |n|/R^2$).

Introduce now the two-component vector:

$$\boldsymbol{z} \equiv \left\{ z, \frac{R}{\sqrt{|n|}} z' \right\},$$

(there is actually no particular reason for choosing the coefficient of z' in this fashion: nothing would be changed in the sequel by taking a different coef-

(3) M. SANDS and B. TOUSCHEK: *Nuovo Cimento*, **10**, 604 (1953).

ficient, except the discussion of Part 3·1, which should have to be modified in an obvious manner). If z_0 denotes the value of z at entry of a section, one sees immediately from (1) and (2) that the value at exit of that section is given by:

$$(3) \qquad z = \Gamma_+(l_+)z_0\,, \qquad \text{with} \qquad \Gamma_+(l_+) = \begin{pmatrix} \cos\frac{\sqrt{|n|}}{R}\,l_+ & \sin\frac{\sqrt{|n|}}{R}\,l_+ \\ -\sin\frac{\sqrt{|n|}}{R}\,l_+ & \cos\frac{\sqrt{|n|}}{R}\,l_+ \end{pmatrix}$$

for a fs of length l_+ and by

$$(4) \qquad z = \Gamma_-(l_-)z_0\,, \qquad \text{with} \qquad \Gamma_-(l_-) = \begin{pmatrix} \cosh\frac{\sqrt{|n|}}{R}\,l_- & \sinh\frac{\sqrt{|n|}}{R}\,l_- \\ \sinh\frac{\sqrt{|n|}}{R}\,l_- & \cosh\frac{\sqrt{|n|}}{R}\,l_- \end{pmatrix}$$

for a dfs of length l_-. It is also readily verified that, at exit of a ffs of length l_0

$$z = \Gamma_0(l_0)z_0\,, \qquad \text{with} \qquad \Gamma_0(l_0) = \begin{pmatrix} 1 & \frac{\sqrt{|n|}}{R}\,l_0 \\ 0 & 1 \end{pmatrix}.$$

A convenient property of these matrices is that

$$(5) \qquad \det\Gamma_+ = \det\Gamma_- = \det\Gamma_0 = +1\,.$$

If the particle enters a sequence of sectors, say, dfs + ffs + fs + ffs′, it exits from it with

$$z = \Gamma_0(l_0')\Gamma_+(l_+)\Gamma_0(l_0)\Gamma_-(l_-)z_0\,.$$

The effect of crossing any number of sectors of various types is always described by a matrix which is the product (from right to left) of the matrices corresponding to the crossed sectors (from left to right).

The machine is a periodic structure, in which any given type of sector reappears again and again. We can consider it therefore as composed of consecutive identical « units », calling by this name the *smallest* set of contiguous sectors which can be regarded as the periodic element.

We leave, otherwise, completely arbitrary the number and type of sectors which, together, form the unit of the machine. We know from the preceding considerations that the unit is characterized by a 2×2 matrix A with real

elements, such that $\boldsymbol{z} = \Delta \boldsymbol{z}_0$ is the exit value of $\boldsymbol{z}$ for a particle which has entered the unit with $\boldsymbol{z} = \boldsymbol{z}_0$. This is all we need to discuss most of the properties of the accelerator.

2·2. – Call $2x$ the spur of the matrix Δ. Since Δ is a product of matrices Γ_+, Γ_-, Γ_0, it follows from (5) that also $\det \Delta = +1$. The characteristic equation of Δ is therefore:

$$\lambda^2 - 2x\lambda + 1 = 0 . \tag{6}$$

Consider the roots λ_1, λ_2 of (6). Three cases are possible:

1) λ_1 and λ_2 are complex conjugate, therefore of the form

$$\lambda_1 = e^{i\varphi} , \quad \lambda_2 = e^{-i\varphi} ; \quad \varphi \neq k\pi , \quad x = \cos\varphi . \tag{7}$$

The conditions for this to happen is $|x| < 1$.

2) $\lambda_1 = \lambda_2$ real: $= 1$ if $x = 1$, $= -1$ if $x = -1$.

3) $\lambda_1 \neq \lambda_2$, both real. This happens if $|x| > 1$.

The following theorem holds (for the proof, see App. I): *The necessary and sufficient condition for the existence of stable orbits is* $|x| < 1$, i.e. that (6) have complex conjugate roots. The following corollary is then immediate: *A necessary condition for stability is that* Δ *be not symmetric* (or else, Δ being in this case Hermitean because of its reality, its eigenvalues would be real).

The stated theorem tells whether the machine allows stable orbits or not; it does not suffice, however, to delimit the bundle of orbits which are actually stable. (In other words, the initial conditions for which a particle, injected into the (ideal) machine, does remain at all times within the vacuum chamber). Information on this point, which is clearly of importance, can be had if one knows:

a) number, type and ordering of the sectors which together form the unit (smallest periodic element of the machine), in order to determine the maximum value that $\boldsymbol{z}$ can reach within the unit, for given initial conditions. This maximum depends upon the geometry of the unit. We outline in Part 3·1, the general procedure by treating in detaail simple example.

b) the maximum value that $\boldsymbol{z}$ can reach after crossing an arbitrarily large number of sectors. This equation is treated in the next section.

2·3. – The only situation of interest is that described by (7). The discussion is facilitated by observing that Δ can be decomposed as follows:

$$\Delta = \cos\varphi + J \sin\varphi \tag{8}$$

(where J — which has, clearly, spur zero — can be immediately constructed as $(\Delta - \cos\varphi)/\sin\varphi$).

J has the remarkable property that:

$$J^2 = -1 \tag{9}$$

(this can be verified by using the fact that Δ must obey the same characteristic equation (6) as its eigenvalues; the same result is derived in App. I from a more pedestrian point of view). As a consequence of (9), we can evaluate any power of Δ as if Δ were a complex number:

$$\Delta^m = e^{mJ\varphi} = \cos m\varphi + J \sin m\varphi \,. \tag{10}$$

We can now answer question b) of the preceding section. Write $m\varphi = \vartheta$ and consider, in the plane spanned by $\boldsymbol{z}$, the curve of equation:

$$\boldsymbol{z} = \cos\vartheta \boldsymbol{z}_0 + \sin\vartheta \boldsymbol{t}_0 \,, \tag{11}$$

where $\boldsymbol{t}_0 = J\boldsymbol{z}_0$ (fig. 1). This is the equation of an ellipse, of which $\boldsymbol{z}_0$ and $\boldsymbol{t}_0$ are two conjugate semidiameters. The explicit knowledge of J permits one to deduce, with simple geometrical considerations which we leave to the reader, direction and magnitude of the great semiaxis as a function of $\boldsymbol{z}_0$, from which the wanted information is readily gathered. We may just mention here that, once $\boldsymbol{z}_0$ and $\boldsymbol{t}_0$ are given, a standard construction due to Chasles permits one to solve this problem graphically (see any textbook on projective geometry); also, if one is interested only in the magni-

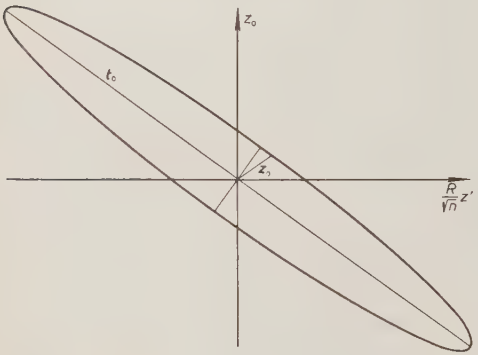

Fig. 1. – Numerical data corresponding to the figure: $|n| = 5\cdot10^3$; $R = 10^4$ cm; $\frac{\sqrt{|n|}}{R} L = \frac{\pi}{2}$, viz $L = 2{,}24\cdot10^2$ cm. $L_0 = 4\cdot10$ cm, viz $h = 2.8\cdot10^{-1}$; it follows $2x = -1{,}586$.

$$J = \frac{1}{0{,}613}\begin{pmatrix} 2{,}21 & 1{,}668 \\ -3{,}1535 & -2{,}21 \end{pmatrix},$$

$$\boldsymbol{z}_0 = \begin{pmatrix} 1 \\ R/\sqrt{|n|}\cdot10^{-2} \end{pmatrix};$$

it follows $|\boldsymbol{z}_0| = 1{,}75$ cm; $|\boldsymbol{t}_0| = 12{,}8$ cm; $\delta_0 = 109°$. In this numerical and graphical example, the great semiaxis differ in magnitude and direction very little from $\boldsymbol{t}_0$. It is not represented in the figure for the sake of clearness.

tude of the semiaxes, the following construction suffices (it is an immediate consequence of Apollonius' theorems on conics): let a and b be the magnitudes of the semiaxes, δ_0 the angle between $\boldsymbol{z}_0$ and $\boldsymbol{t}_0$: draw the parallelogram with sides of length $|\boldsymbol{z}_0|$ and $|\boldsymbol{t}_0|$, the angle between these two sides being taken $= (\pi/2) - \delta_0$: the two diagonals of this parallelogram have length $a+b$ and $a-b$ respectively.

One sees from (10) that, *for arbitrary m and a stable orbit, all points $\Delta^m \boldsymbol{z}_0$ lie on the ellipse* (11). The following theorem is also evident:

The necessary and sufficient condition for all bounded orbits to have a period of m units is $\Delta^m = +1$, i.e.: from (7) and (10)):

$$\varphi = \frac{2k}{m}\pi\,, \qquad \frac{2k}{m} \neq \text{integer}\,, \tag{12}$$

which gives distinct roots only for $0 < 2k < m$. If m is not a prime one finds, of course, also the periodicity conditions relative to its prime factors: thus, for $m = 6$, one finds $\varphi_1 = \pi/3$ and $\varphi_2 = 2\pi/3$, the second being the periodicity condition for $m = 3$. Periodicity is forbidden for $n = 1$ (see App. I), and then also for $n = 2$.

Assume now that one wants to build a machine such that all orbits have a period of m (and not less than m):

1) a value of φ must be obtained from (12) which secures this periodicity (and not less). This sets one condition on the geometry of the unit, by fixing $x = (\Delta_{11} + \Delta_{22})/2 = \cos\varphi$;

2) a second condition is given by assigning the total number of fs, dfs and ffs contained in the machine.

These two condition may or may not suffice to determine uniquely the geometry. A very simple pattern is that corresponding to a unit dfs + ffs + fs + + ffs, with the dfs and fs having both length L, and the ffs's having all length L_0. The two stated conditions suffice in this case to determine uniquely the geometry, i.e. the values of L and L_0. Δ is given in this case, from (3)-(5), by (writing: $\cos(\sqrt{|n|}\,L/R) = a$, $\cosh(\sqrt{|n|}\,L/R) = a'$, $(\sqrt{|n|}\,L_0/R) = h$, $b = \sqrt{1-a^2}$, $b' = \sqrt{a'^2-1}$)

$$\begin{aligned}
\Delta_{11} &= -bb'h^2 + (2ab' - ba')h + (aa' + bb')\\
\Delta_{12} &= -ba'h^2 + (2aa' - bb')h + (ab' + ba')\\
\Delta_{21} &= -bb'h + (ab' - ba')\\
\Delta_{22} &= -ba'h + (aa' - bb')
\end{aligned}$$

and

$$2x = -bb'h^2 + 2(ab' - ba')h + 2aa'\,.$$

We record in App. II, for possible future reference, some formulae which give an alternative expression of Δ^m.

3·1. – We study here in detail a very simple example which shows clearly the role of the fs's and dfs's and indicates the procedure to follow in general to answer question *a*) of 2·2.

Consider the unit as composed only of one dfs of length L and one fs of length $(\pi/2)(R/\sqrt{|n|})$, so that

$$\Delta = \Gamma_+\left(\frac{\pi R}{2\sqrt{|n|}}\right)\Gamma_-(L) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \cosh\dfrac{\sqrt{|n|}\,L}{R} & \sinh\dfrac{\sqrt{|n|}\,L}{R} \\ \sinh\dfrac{\sqrt{|n|}\,L}{R} & \cosh\dfrac{\sqrt{|n|}\,L}{R} \end{pmatrix}.$$

One finds immediately that $\Delta^2 = -1$, *regardless of the value of* L. Periodicity occurs therefore after four units, i.e. eight sectors. We want to determine the initial conditions for which a particle remains bounded in the vacuum chamber.

Call d the half-aperture of the dfs. Boundedness requires that $|z| < d$ in the dfs. Assume that

$$|z_0| < k, \qquad \left|\frac{Rz_0'}{\sqrt{|n|}}\right| < k. \tag{13}$$

Since

$$z \leqslant z_0\left|\cosh\frac{\sqrt{|n|}\,L}{R}\right| + \left|\frac{Rz_0'}{\sqrt{|n|}}\right|\sinh\frac{\sqrt{|n|}\,L}{R} < k\left(\cosh\frac{\sqrt{|n|}\,L}{R} + \sinh\frac{\sqrt{|n|}\,L}{R}\right),$$

one finds the condition

$$k \leqslant \frac{d}{\cosh\dfrac{\sqrt{|n|}\,L}{R} + \sinh\dfrac{\sqrt{|n|}\,L}{R}}. \tag{14}$$

In the next fs, from (3):

$$|z| < d(|\cos\vartheta| + |\sin\vartheta|) \leqslant \sqrt{2}\,d. \tag{15}$$

If the fs has a half aperture $D \geqslant \sqrt{2}\,d$, all the orbits (13), (14) remain bounded (clearly, once boundedness is insured in Δ, it is insured throughout, since,

here, $\Delta^2 = -1$). The role of the dfs is to fix k by means of (14); thus on finds, with $d \leqslant D/\sqrt{2}$:

$$\frac{\sqrt{|n|}L}{R} = \frac{\pi}{2} \qquad k = \frac{d}{4.8},$$

$$\text{»} \quad = \frac{3\pi}{4} \qquad k = \frac{d}{10.5},$$

$$\text{»} \quad = \pi \qquad k = \frac{d}{23.1}.$$

3·2. – We add a few final remarks to the preceding considerations.

1) It is possible to describe by means of matrices of type Γ also the sectors in which an electric field is applied to accelerate the particles. The net action of these sectors would immediately show up, by doing so, as a damping effect. We forego this discussion, since stability should be secured independently of the existence of accelerating fields.

2) It is not essential to the preceding treatment that the magnetic field gradient $|n|$ have the same value for all fs's and dfs's. The generalization would be quite immediate, consisting essentially in slight changes in notation. Also, it is not essential that $\det \Gamma = 1$, $\det \Delta = 1$.

3) The example considered in the previous section shows that the dfs's might be given, in principle, an aperture $2d$ smaller than that $2D$ of the fs's the ratio D/d depending upon the geometry of the machine.

4) From (3), (4) and (5) one readily gathers that the matrices Δ can be given a simple geometrical interpretation in the plane of fig. 1. $\Gamma_+(l_+)$ rotates the vector $\boldsymbol{z}_0$ by an angle $\sqrt{|n|}\, l_+/R$ in the positive direction, the maximum value of z in the fs being given by the maximum z reached in the rotation. $\Gamma_-(l_-)$ causes the end point of $\boldsymbol{z}$ to move along a hyperbola (a simple construction could be likewise found); $\Gamma_0(l_0)$ adds to $\boldsymbol{z}_0 \equiv \{z_0,\ Rz_0'/\sqrt{|n|}\}$ the vector $\{l_0 z_0', 0\}$. A detailed study of this fact, in conjunction with the considerations made with regard to fig. 1, would afford a quite simple means of determining graphically the trajectory of a particle entering the vacuum chamber with given initial conditions.

In conclusion, the authors wish to thank Prof. E. Amaldi for his constant encouragement in this work, and him, Drs. M. Sands and B. Touschek, for many interesting discussions.

Appendix I.

Proof of Theorem of 2·2.

We start by considering case 3): $\lambda_1 \neq \lambda_2 = 1/\lambda_1$, both real. Let $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ denote the (real) eigenvectors of Δ. Any vector $\boldsymbol{z}$ can be expressed as a linear combination of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$:

$$\boldsymbol{z} = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 .$$

It follows:

$$\Delta^m \boldsymbol{z} = \alpha_1 \lambda_1^m \boldsymbol{u}_1 + \alpha_2 \frac{1}{\lambda_1^m} \boldsymbol{u}_2$$

which proves the theorem in this case.

Consider now case 2): $\lambda_1 = \lambda_2 = \pm 1$, for $x = \pm 1$. Two cases are possible: *a*) Δ is not symmetric: then it can be reduced, by transformation with a suitable matrix, at most to the form $\begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}$, from which one readily sees that the theorem is again true; *b*) Δ is symmetric and can be reduced therefore to the form ± 1. It is not possible to discard this case a priori by considering exclusively the behaviour of the ideal machine. The argument that leads to its rejection is that the condition $x = \pm 1$ is very critic, because small deviations from this value due to the unavoidable imperfections of the actual machine might lead to instability (case (3)). See also ref. ([3]).

To study case 1): $\lambda_1 = \bar{\lambda}_2 = e^{i\varphi}$, $x = \cos\varphi \neq \pm 1$, we express again $\boldsymbol{z}$ as a l.c. of the eigenvectors $\boldsymbol{u}$ and $\bar{\boldsymbol{u}}$ (with complex coefficients): then

$$\boldsymbol{z} = \alpha \boldsymbol{u} + \bar{\alpha}\, \bar{\boldsymbol{u}},$$

$$\Delta^m \boldsymbol{z} = \alpha \exp[im\varphi]\boldsymbol{u} + \bar{\alpha} \exp[-im\varphi]\bar{\boldsymbol{u}} = \cos m\varphi \boldsymbol{z} + \sin m\varphi \boldsymbol{t} ,$$

with:

$$\boldsymbol{t} = i(\alpha \boldsymbol{u} - \bar{\alpha}\, \bar{\boldsymbol{u}}) = \frac{\Delta - \cos\varphi}{\sin\varphi} \boldsymbol{z} \quad \text{(from the above, for } m = 1);$$

setting $J = (\Delta - \cos\varphi)/\sin\varphi$, comparison of

$$\Delta^m = \cos m\varphi + J \sin m\varphi$$

with the analogous expression for complex numbers, shows that $J^2 = -1$. (This method of proof could be applied also to case 3), but we need not insist on this point).

Appendix II.

Alternative study of $\Delta^m = 1$ for $|x| < 1$.

Write

$$\gamma = 2x = 2\cos\varphi\,.$$

From:

$$\Delta^2 - \gamma\Delta + 1 = 0\,,$$

one finds by recursion:

$$\Delta_m = \begin{pmatrix} P_{m-1}(\gamma)\Delta_{11} - P_{m-2}(\gamma) & P_{m-1}(\gamma)\Delta_{12} \\ P_{m-1}(\gamma)\Delta_{21} & P_{m-1}(\gamma)\Delta_{22} - P_{m-2}(\gamma) \end{pmatrix},$$

where

$$P_0(\gamma) = 1\,,$$

$$P_1(\gamma) = \gamma\,,$$

$$P_m(\gamma) = \gamma P_{m-1}(\gamma) - P_{m-2}(\gamma) = \begin{vmatrix} \gamma & 1 & 0 & 0 & \dots & 0 \\ 1 & \gamma & 1 & 0 & \dots & 0 \\ 0 & 1 & \gamma & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \gamma \end{vmatrix},$$

the determinant being of order m. The condition for periodicity after m units $\Delta^m = +1$ is

$$P_{m-1}(\gamma) = 0\,, \qquad P_{m-2}(\gamma) + 1 = 0\,.$$

The roots common to these two equations can be shown to be all and only the roots of:

$$(m = 2k) \qquad P_{k-1}(\gamma) = 0$$
$$(m = 2k+1) \qquad P_{k-1}(\gamma) + P_k(\gamma) = 0\,.$$

They coincide with (12), as is seen from the relation:

$$P_k(2\cos\varphi) = \frac{\sin(k+1)\varphi}{\sin\varphi}\,.$$

RIASSUNTO

Viene indicato un semplice metodo per lo studio delle proprietà di un acceleratore a « focalizzazione forte » di struttura arbitraria. Si mostra come da esso derivino prontamente le condizioni necessarie e sufficienti per la stabilità e per la periodicità (con periodo assegnato a piacere) delle orbite. Si danno infine criteri che consentono di determinare il campo dei valori iniziali in corrispondenza dei quali le orbite risultano stabili.

Alignment Errors in the Strong-Focusing Synchrotron.

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Summary. — A method is presented for the treatment of errors in alignment between the sectors of a strong focusing accelerator. Motion in one lateral coordinate only is treated, and the individual sectors are assumed to be ideal. It is found that if the ideal orbits (without misalignments) are periodic in one revolution the effect of the misalignments is to produce unbounded orbits, but that away from such resonances stable orbits exist. Estimates are made of the alignment tolerances required for reasonable amplitudes of the perturbed motion. The essential features of the results are independent of the detailed nature of the guide field so long as the ideal orbits are stable (**).

1. – Introduction.

Courant, Livingston and Snyder [1] have proposed a principle for high energy synchrotrons in which the strong-focusing property of high-gradient fields is employed to produce stable orbits within a small aperture. They have derived design parameters for certain ideal configurations of the magnetic guide field. Departures of the field from the assumed form will produce distortions in the ideal orbits. Since the same field errors will be felt by the

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(**) *Note added in proof.* — After the circulation of this work in manuscript form, Mr. F. K. Goward kindly afforded to us a manuscript by J. S. Bell, which is to be published by Her Britannic Majesty's Stationary Office. Bell, using a method similar to ours, has treated all the problems discussed in the present paper.

(1) E. Courant, S. Livingston and H. Snyder: *Phys. Rev.*, **88**, 1190 (1952).

particle in every revolution, it is possible that their effects may add coherently and give rise to large distortions of the orbits. We shall discuss these resonance effects for field errors of a particular type.

We consider a generalised design of a strong focusing guide field which consists of a large number of discrete sectors. By a sector we mean a region in which the field gradient is independent of the azimuth. In a sector this gradient may be positive or negative, or the field may be zero. We assume for convenience that the complete guide field consists of a sequence of M periodic elements, each periodic element consisting of a prescribed sequence of sectors. We assume, further, that the prescribed field gives a single valued equilibrium orbit and permits stable lateral oscillations about this orbit. We treat only the case in which the vertical and radial oscillations are independent. We consider, therefore, only motion in the vertical direction. With a change in the values of the parameters the same results would be found for the radial component of the motion. Also, we make the usual adiabatic approximation, ignoring effects due to the increasing magnetic field and to the acceleration of the particles.

In the following sections we assume that we are given the specifications of a particular guide field, and consider the nature of the orbits which occur in a field constructed from these specifications, but with errors in alignment between adjacent sectors. From these considerations we find that restrictions on the design are required if the perturbed orbits are to remain stable. We find also the relation between the expected amplitudes of the perturbed orbits and the alignment tolerances.

2. – The Equations of Motion.

In each sector the vertical displacement of the particle from its equilibrium orbit is assumed to be governed by the relation

$$\frac{\mathrm{d}^2 z}{\mathrm{d}l^2} + \omega_j^2 z = 0 \,, \tag{1}$$

where l is the distance along the equilibrium orbit. The parameter ω_j is a characteristic of the j-th sector and may be real (defocusing), imaginary (focusing) or zero (field-free).

CAIANIELLO (2) has pointed out that it is convenient to represent the « state » of a particle by the two component symbol

$$\boldsymbol{z} = \begin{pmatrix} z \\ k \dfrac{\mathrm{d}z}{\mathrm{d}l} \end{pmatrix},$$

(2) E. R. CAIANIELLO: *Nuovo Cimento*, **10**, 581 (1953).

where k is an arbitrary length. Then the value of this state vector $\boldsymbol{z}_j$ at the end of the j-th sector can be expressed in terms of its value at the entrance to the sector (and thus at the exit of the preceding sector) by

$$\boldsymbol{z}_j = \Gamma_j \boldsymbol{z}_{j-1} , \tag{2}$$

where Γ_j is a matrix:

$$\Gamma_j = \frac{1}{2} \begin{vmatrix} \exp[\omega_j L_j] + \exp[-\omega_j L_j] & \frac{1}{k\omega_j}(\exp[\omega_j L_j] - \exp[-\omega_j L_j]) \\ k\omega_j(\exp[\omega_j L_j] - \exp[-\omega_j L_j]) & (\exp[\omega_j L_j] + \exp[-\omega_j L_j]) \end{vmatrix} , \tag{3}$$

with L_j the length of the sector.

Similary we may denote by $\boldsymbol{z}_m$ the state vector at the exit of the m-th periodic element of the guide field. We then have that

$$\boldsymbol{z}_m = \Delta(\boldsymbol{z}_{m-1} + \boldsymbol{p}_{m-1}) , \tag{4}$$

where the matrix Δ is a product of the matrices Γ for the individual sectors which make up the periodic element and where $\boldsymbol{p}_{m-1}$ represents the errors — lateral and angular — in alignment at the entrance to the m-th element. The matrix for the periodic element will have the form

$$\Delta = \begin{matrix} a & b/k \\ kc & d \end{matrix} , \tag{5}$$

where a, b, c, d are real numbers and where necessarily

$$ad - bc = 1 . \tag{6}$$

If a particle is injected into the first element with a state vector $\boldsymbol{z}_0$, it will arrive again at the first section after one revolution of M elements. Its statevector will then be

$$\boldsymbol{z}(1) = \Delta^M \boldsymbol{z}_0 + \sum_{m-1}^{M} \Delta^{M-n} \boldsymbol{p}_{m-1} , \tag{7}$$

which for convenience we may rewrite in the form

$$\boldsymbol{z}(1) = D\boldsymbol{z}_0 + \boldsymbol{p}(1) , \tag{8}$$

where $D = \Delta^M$ is the design characteristic of the complete guide field and $\boldsymbol{p}(1)$ is a characteristic of all the alignment errors of the machine.

Repeated application of equation (8) shows that after n complete revolutions the state vector $\boldsymbol{z}(n)$ of the particle is given by

$$\boldsymbol{z}(n) = D^n \boldsymbol{z}_0 + \sum_{j=1}^{n} D^{j-1} \boldsymbol{p}(1) \,. \tag{9}$$

The first term represents the stable oscillations in an ideal machine, the second the departures from the stable orbits caused by the alignment errors.

In the following section we examine in more detail the distortion term in equation (9) for a given $\boldsymbol{p}(1)$. We return later to the evaluation of $\boldsymbol{p}(1)$ in terms of the alignment errors.

3. – Coherent Addition of Distortion.

The cumulative effect of orbit distortions after n revolutions

$$\boldsymbol{p}(n) = \sum_{j=1}^{n} D^{j-1} \boldsymbol{p}(1) \,, \tag{10}$$

evidently depends in a critical way upon the *ideal* (design) periodicity of the orbits. If, for example, the ideal orbits have a period of one revolution, or an integral submultiple — as in the design suggested in reference [1] — then D is the unit matrix. The error effects add coherently and the total distortion after n revolutions is $n\boldsymbol{p}(1)$ and the perturbed orbits are not bounded unless $\boldsymbol{p}(1) = 0$.

By a slight modification of such a design the matrix D may be chosen to be -1. In this case there is destructive interference and the cumulative distortion is $\frac{1}{2}(1 - (-1)^n)\boldsymbol{p}(1)$ and the orbits are bounded. It may be emphasized here that the error vectors are not assumed to be small and may, in fact, have a displacement component which is an appreciable fraction of the aperture. In such a situation the actual periodicity of the orbits may be far off the ideal one, but the validity of our conclusions is not altered.

To treat the case of a general D, we notice that the matrices of the periodic element define eigenvectors and associated eigenvalues by the relation

$$A\boldsymbol{v} = \lambda \boldsymbol{v} \,. \tag{11}$$

From the characteristic equation of (11) and by using equation (5) for the representation of the matrix A one finds that the eigenvalues are given by

$$\lambda = \tfrac{1}{2}(a + d) \pm (\tfrac{1}{4}(a + d)^2 - 1)^{1/2} \,, \tag{12}$$

where we have made use of the condition (6). The requirement that the unperturbed field give stable orbits necessarily implies (cf. Reference [2]) that $|a+d|<2$. In this case the eigenvalues (12) have the form

$$\lambda_1 = \exp[i\varphi]\,; \qquad \lambda_2 = \exp[-i\varphi]\,, \tag{13}$$

where $\cos\varphi = \frac{1}{2}(a+d)$ and we take $0<\varphi<\pi$. Since the elements of A are real, the components of the eigenvectors are in general complex and it is clear that the two eigenvectors can be chosen to be complex conjugates.

We may now take $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ to be the eigenvector whose eigenvalue is λ_1. From equation (11) the two components of the eigenvector are related by

$$\frac{v_2}{v_1} = -\frac{k}{b}(a-\lambda_1)\ . \tag{14}$$

If we adopt as a normalisation condition that $v_1 v_2 = 1$ then v_1 and v_2 are given by

$$v_2^2 = \frac{1}{v_1^2} = -\frac{k(a-\lambda_1)}{b}\ . \tag{15}$$

We now recall that k is an arbitrary (real) length which establishes the magnitude of the 2nd component of the state vector. It simplifies the subsequent manipulations if k is now specified so that the absolute value of v_2 and hence of v_1 is unity. Thus k will depend on the design parameters and is in fact

$$k = \frac{b}{\sqrt{1-ad}}\ . \tag{16}$$

With this choice of k the components of the eigenvector become

$$v_1 = \exp[i\psi]\,, \qquad v_2 = \exp[-i\psi]\,, \tag{17}$$

where ψ is the angle between 0 and $\frac{1}{2}\pi$ given by

$$\sin 2\psi = \frac{\sin\varphi}{\sqrt{1-ad}}\ . \tag{18}$$

It follows from the definition $D = A^M$ that its eigenvectors are also $\boldsymbol{v}$ and $\boldsymbol{v}^*$ with eigenvalues $\exp[\pm i\Phi]$ where $\Phi = M\varphi$.

Returning now to the evaluation of $\boldsymbol{p}(n)$ we may express $\boldsymbol{p}(1)$ in terms

of the eigenvectors $\boldsymbol{v}$ and $\boldsymbol{v}^*$ by

$$\boldsymbol{p}(1) = \gamma\boldsymbol{v} + \gamma^*\boldsymbol{v}^*\,, \tag{19}$$

where, clearly, the second coefficient must be the complex conjugate of the first if the sum is to be real. Substituting from equation (19) in (10) and evaluating the sum — a geometrical series — we find that

$$\boldsymbol{p}(n) = \frac{\exp[in\Phi]-1}{\exp[i\Phi]-1}\,\gamma\boldsymbol{v} + \text{comp. conj.}\,.$$

Using equation (19) to evaluate the magnitude of γ in terms of the components of $\boldsymbol{p}(1)$, we find that the first component $p_1(n)$ of $\boldsymbol{p}(n)$ is

$$p_1(n) = \tag{20}$$

$$= \frac{\sin \tfrac{1}{2}n\Phi}{\sin \tfrac{1}{2}\Phi}\left(\frac{p_2^2(1)+p_1^2(1)+2p_2(1)p_1(1)(\sin^2\psi-\cos^2\psi)}{\sin^2 2\psi}\right)^{1/2}\cos(\Theta+\psi+\tfrac{1}{2}(n-1)\Phi),$$

where Θ is the argument of γ. The same expression is found for $p_2(n)$ with ψ replaced by $-\psi$.

The first factor of (20) depends only on the design periodicity of the orbits and exhibits the resonance behaviour described at the beginning of this section. It is evident that if the angle Φ is near an integral multiple of 2π, the distortion of the first revolution will in general be amplified on successive revolutions, reaching eventually a maximum amplitude many times that of the first revolution. If on the other hand Φ is not too close to a multiple of 2π the accumulated amplitude of the distortion will oscillate about zero with a maximum amplitude not too different from that of the first revolution. In any case the maximum amplitude of the distortion will be near the upper bound given by

$$|p_1(n)| < \left|\frac{\sin\frac{n}{2}\Phi}{\sin\frac{1}{2}\Phi}\right|\frac{|p_1|+|p_2|}{\sin 2\psi}\,, \tag{21}$$

and an identical upper bound is obtained for $p_2(n)$.

We may emphasize that the possibility of resonance is determined by the value of Φ which is in turn a rapidly varying function of φ. The range of values of φ (or in the notation of reference [1], of $\sqrt{n_2}/N$) in which stable ideal orbits are possible is therefore punctuated at small intervals by « reso-

nance points » — roughly $\frac{1}{2}M$ in number — at which the alignment errors may lead to instability.

The factor $1/\sin 2\psi$ in (21) depends on the details of the periodic element of the guide field. It is, in fact, a form-factor of the ideal orbits which relates the maximum of their displacement to the average value. We have evaluated the factor $1/\sin 2\psi$ for a guide field similar to that proposed in the design of reference (1). We take for the periodic element a focusing sector and a defocusing sector each of the same length L, and with field gradients equal to $-n_1$ and n_1 respectively. No field free sectors are included. Then in equation (3) the parameters $\omega_j L_j$ become $2\pi\sqrt{\pm n_1}/N$. In Fig. 1 we plot $1/\sin 2\psi$ as a function of n_1/N^2. It is seen to be near 2 except for a rapid increase near the limit of stability for the ideal orbits. The nature of the factor $1/\sin 2\psi$ will not be very different for other reasonable choices of a periodic element.

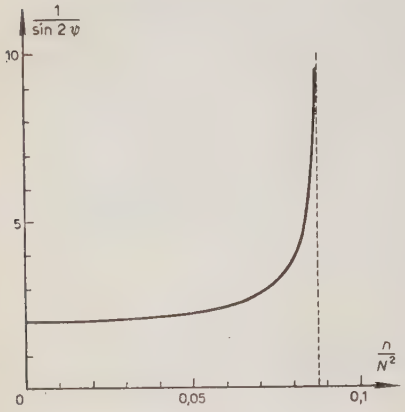

Fig. 1. The form factor $1/\sin 2\psi$ plotted against $|n|/N^2 = |\omega^2| L/4\pi^2$ for a guide field containing alternate focusing and defocusing sectors of the same length and absolute value of gradient.

4. – Cumulative Distortion in the First Revolution.

We return now to the consideration of the quantity $\boldsymbol{p}(1)$ defined by equations (7) and (8). The errors $\boldsymbol{p}_m$ may be systematic (if due, for example, to necessary compromises in the mechanical structure). In this cas $\boldsymbol{p}(1)$ can be evaluated from the properties of the errors and of the ideal orbits defined by the matrix Δ. We shall not treat this case any further, but only mention that, if the ideal orbits have a short cycle periodicity (many periods in one revolution) the most serious systematic alignment errors will be those which have the same period as the orbit oscillations. Since the shortest orbit periodicity is likely to have a wavelength of several periodic elements of the field, the above resonance is not likely to arise in practice.

If the individual alignment errors which give rise to $\boldsymbol{p}(1)$ are random with a normal distribution we may estimate the magnitude of $\boldsymbol{p}$ (1). We take the mean value of both components of $\boldsymbol{p}_m$ to be zero, and the mean square value of the two components of $\boldsymbol{p}_m$ to be δ_1^2 and δ_2^2 respectively. Applying the

methods of Section 3 and averaging we find that the mean square value of the displacement component $p_1(1)$ $\boldsymbol{p}(1)$ is

$$\overline{p_1^2} = \frac{1}{2}(\delta_1^2 + \delta_2^2)\sum_{m=0}^{M-1}\left\{\frac{\cos^2(\psi + m\varphi)}{\cos^2\psi} + \frac{\sin^2(\psi + m\varphi)}{\sin^2\psi}\right\} - \tag{22}$$

$$-\frac{1}{2}(\delta_1^2 - \delta_2^2)\sum_{m=0}^{M-1}\frac{\cos(\psi + m\varphi)\sin(\psi + m\varphi)}{\cos\psi\sin\psi},$$

and a similar expression is obtained for $\overline{p_2^2}$. In the evaluation of equation (22) we have neglected the constraint on the error $\boldsymbol{p}_m$ which arises from the requirement that the sum over the errors must be zero around a full circuit. If M is large the terms neglected are about $1/M$ of the terms retained. It may also be pointed out that our assumption that the errors in the two components of $\boldsymbol{p}_m$ are independent may be violated by certain assembly procedures, but again the results will not be likely to be very different.

If the number of periodic elements is large and if the characteristic angle φ is not zero or π (limits of the stability region) we may replace the quantities under the summation signs by their average values. We find that in this approximation

$$p_1^2 \approx p_2^2 \approx \frac{1}{2}(\delta_1^2 + \delta_2^2)\frac{M}{\sin^2 2\psi}.$$

It is seen that the combined effect of the alignment errors increases as $\sqrt{M}$ as might be expected (*). The factor $1/\sin 2\psi$ is the same form-factor which appears in section 3.

The factor M in equation (23) is the number of periodic elements. It can be shown that if we also allow the same type of alignment errors at each sector, within the periodic element the result is not very different from (23) with M replaced by N' where N' is the total number of sectors which may have alignment errors. (Field-free sectors must be excluded from this number since they, obviously, cannot have alignment errors).

We may try now to give a quantitative estimate of the tolerances required for the alignment of adjacent sectors. If we specify constructional tolerances so that lateral displacement errors and angle errors contribute equally to the amplitude of the distortion (i.e. $\delta_1 = \delta_2$) and if P is the maximum amplitude of the displacement which we can allow in the first revolution we have from

(*) In the course of this work we have had occasion to see a manuscript note by J. D. Lawson who reaches a similar conclusion from different arguments.

equation (23)

$$\delta_1 = \delta_2 \leqslant \frac{P \sin 2\psi}{\sqrt{N'}} \,. \tag{24}$$

We take again as an example the design suggested in Reference (1) (assumed to be modified slightly to avoid the resonance discussed in section 3) neglecting the straight sections. For this case $N^2/n_1 = 16$, $N = N' = 240$, $L = 240$ cm. We also take $P = 0.6$ cm (rather large). With these values

$$\delta_1 = \delta_2 \leqslant 1.5 \cdot 10^{-2} \text{ cm} \,.$$

The angle associated with δ_2 is $\varepsilon = \delta_2/k$. For the guidefield assumed above

$$k = 145 \text{ cm} \approx \frac{\sqrt{3}}{4\pi} \frac{n_1}{N^2} L \,.$$

The angular tolerance is then $\varepsilon = 1.1 \cdot 10^{-4}$. A more pertinent number is perhaps $\varepsilon L = 2.5 \cdot 10^{-2}$ cm.

In view of the smallness of these tolerances it will undoubtedly be found expeditious to place adjustable correcting perturbations at one or more places along the orbit which could be set empirically to cancel the cumulative effects of the random errors.

5. – General Remarks and Qualifications.

We have not treated here the problem related to the variations of the internal parameters of the individual sectors — field-gradient, length, average field, etc. It is evident that the considerations of section 3 will lead to the specification of tolerances on the effective lengths and gradients of the individual sectors such that the cumulative effect of such errors cannot shift the value of the characteristic angle from an antiresonance value to a resonance. It would appear that the problem of eddy currents and permeability changes in the magnet structure must be examined with regard to the manner in which they affect Φ. The great danger would seem to be that changes of the field gradient during the acceleration cycle will force one to pass through one of the resonance values of Φ.

Perhaps the strongest limitation to the applicability of our conclusions rests with the assumption of linearity, that the restoring force constant ω_j in equation (1) does not depend on the amplitude of z. It is possible that in a practically realizable field the nonlinearities will be so great as to destroy completely the coherence of the perturbations on successive revolutions. It

seems, however, likely that small nonlinearities can be tolerated without invalidating the essential qualitative aspects of our results.

We envisage the possibility of expressing other constructional errors in terms of equivalent alignment errors. For example a lag in time of the field in one sector is equivalent to a small change in the constant φ associated with that sector and a lateral displacement. This latter effect can be treated by the methods used here. CAIANIELLO has suggested that small non-linear effects may also be treated as effective alignemt errors. In this case, however the « alignment errors » are not a fixed property of the guide field and the method of Part 3 cannot be applied. It is possible that this problem can be treated approximately by the methods developed in Part 4 applied to many revolutions.

We are grateful to Dr. E. CAIANIELLO for many stimulating discussions and to Prof. E. AMALDI for his constant encouragement.

RIASSUNTO

Si dà un metodo per lo studio degli errori di allineamento tra i settori di un acceleratore a « focalizzazione forte ». Si studia il moto in una scla coordinata, assumendo i singoli settori idealmente perfetti. Si trova che, se le orbite ideali (non affette da errori) sono periodiche in una sola rivoluzione, gli effetti di tali errori si cumulano fino a produrre instabilità; esistono tuttavia regioni di stabilità. Vengono inoltre valutate le tolleranze necessarie per ottenere ampiezze non eccessive del moto perturbato. I resultati essenziali sono indipendenti dai dettagli della geometria e del campo magnetico fintantochè le orbite ideali sono stabili.

A Covariant Formulation of the Non-Adiabatic Method for the Relativistic Two-Body Problem (II).

M. Cini

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(ricevuto il 14 Marzo 1953)

Summary. — It has been shown in Part I of this work how the Tamm-Dancoff non adiabatic treatment of the relativistic problem of two particles interacting through a neutral pseudoscalar meson field can be put in a covariant form. The integral equations obtained involve infinite kernels corresponding to radiative processes. In the present Part II the techniques of Dyson and Matthews for the renormalization of the S-matrix have been applied in order to separate from the kernel of the integral equations a convergent part, showing that the divergent contributions can be absorbed in unobservable mass and coupling constant renormalization. If we assume the normalized masses and charge to be defined by the same formal series with divergent constant terms as in the S-matrix, as required if a physical meaning is to be attached to the renormalization procedure, finite contributions arise also from self energy kernels of non interacting particles. The terms arising from disconnected loops in the vacuum can be eliminated if the same approximation is used as in the S-matrix theory, namely that the time dependence of the state vector can be neglected in these integrals.

1. – Introduction.

In the first part of this work [1] the Tamm-Dancoff (T.D.) non adiabatic treatment of the relativistic system of two particles interacting through a neutral pseudoscalar meson field has been put in a covariant form. Integral equations up to order G^2 and G^4 have been deduced for the function $\mathcal{A}_\sigma(x', x'')$ defined in (I.7), representing the probability amplitude that the state vector

[1] M. Cini: *Nuovo Cimento*, **10**, 526 (1953), referred to as I.

$\Psi(\sigma)$ describes a free proton and a free neutron at two different space time points x', x'' respectively. These integral equations are, nevertheless, meaningless because the kernels contain, together with finite parts arising from the convergent processes (in the S matrix language) also divergent expressions from the various radiative effects. In this second part it will be shown, by making use of the renormalization procedures worked out recently by FEYNMAN, DYSON and MATTHEWS [2] for the S matrix, how finite contributions can be separated from the divergent kernels and the infinite remainder interpreted as unobservable renormalization of the masses and the coupling constant. A finite and unambiguous integral equation including all radiative effects is thus obtained, at least in principle, up to any order in powers of the coupling constant.

2. – The renormalization of the G^2 equation.

In (I.11) the integral equation to order G^2 with finite kernel is given, but in addition, divergent contributions (I.12), (I.19) and (I.20) have been found. We will leave the discussion of term (I.20), whose kernel arises from a disconnected loop in the vacuum, until later (section **5**). We rewrite (I.12) and (I.19) in the following form:

$$(1)\quad \frac{G^2}{8}\left\{\int_{-\infty}^{\sigma} \mathrm{d}x_1 \int_{-\infty}^{\sigma_1} \mathrm{d}x_2\, S_{FP}(x_1 - x')\gamma_5^P S_{FP}(x_2 - x_1)\gamma_5^P \Delta_F(x_1 - x_2)\mathcal{A}_{\sigma_2}(x_2, x'') + \right.$$

$$\left. + \int_{-\infty}^{\sigma} \mathrm{d}x_1 \int_{\sigma_1}^{\sigma} \mathrm{d}x_2\, S_{FP}(x_1 - x')\gamma_5^P S_{FP}(x_2 - x_1)\gamma_5^P \Delta_F(x_1 - x_2)\mathcal{A}_{\sigma_1}(x_2, x'')\right\}.$$

We have omitted for brevity the terms of the neutron self-energy. Let us consider, in the kernel of (1), the factor

$$(2)\quad N_P(x_2 - x_1) = \gamma_5^P S_{FP}(x_2 - x_1)\gamma_5^P \Delta_F(x_1 - x_2) =$$

$$= -\frac{1}{(8\pi^4)^2}\gamma_5^P \left\{\int \exp\left[-ip(x_2 - x_1)\right]\mathrm{d}p \int \frac{i\gamma^P(p - k) - m}{(p - k)^2 + m^2}\frac{1}{k^2 + \mu^2}\,\mathrm{d}k\right\}\gamma_5^P .$$

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[2] R. P. FEYNMAN: *Phys. Rev.*, **76**, 749, 769 (1949); F. J. DYSON: *Phys. Rev.*, **75**, 486, 1736 (1949), referred to as DI, DII; P. T. MATTHEWS: *Phil. Mag.*, **41**, 185 (1950), referred to as M.

The second integral, well known in the S matrix theory as the only irreducible proton self energy part, is primitive divergent in Dyson's sense [3] and can be written as

$$(3)\quad \int \frac{\gamma_5^P[i\gamma^P(p-k)-m]\gamma_5^P}{[(p-k)^2+m^2](k^2+\mu^2)}\,\mathrm{d}k = (2\pi)^3[-A+(ip\gamma^P+m)B+(ip\gamma^P+m)S_P^c(p)],$$

where A, B and $S_p^c(p)$ are defined by MATTHEWS [4]. A and B are infinite constants and $S_p^c(p)$ vanishes when p is the energy momentum four vector of a free particle.

Thus we can write

$$(4)\quad N_P(x_2-x_1) = \frac{2}{\pi}A\,\delta(x_1-x_2) + \frac{2}{\pi}B\left(\gamma^P\frac{\partial}{\partial x_2}-m\right)\delta(x_1-x_2) + N_P^c(x_2-x_1)\,,$$

where $N_p^c(x)$ is, obviously the Fourier transform of $(ip\gamma^P+m)S_p^c(p)$, apart from numerical factors. Introducing the first term of (4) in (1) one obtains

$$(5)\quad A\frac{G^2}{4\pi}\int_{-\infty}^{\sigma}\mathrm{d}x_1\,S_{FP}(x_1-x')\mathcal{A}_{\sigma_1}x_1,x'')\,.$$

We must consider at this point the renormalization terms not explicitly written in the interaction Hamiltonian (I.2). They are, according to MATTHEWS [5]

$$(6)\quad -\delta m\sum_{\lambda=P,N}\bar{\psi}_\lambda(x)\psi_\lambda(x) - \frac{1}{2}\delta\mu^2\varphi^2(x) - \delta\lambda\varphi^4(x)\,.$$

In the lowest T.D. approximation defined in I, only the first term contributes, giving

$$(7)\quad -\frac{i\delta m}{2}\int_{-\infty}^{\sigma}\mathrm{d}x_1\,S_{FP}(x_1-x')\mathcal{A}_{\sigma_1}(x_1,x'')\,.$$

It is clear that if we choose

$$(8)\quad \delta m = \frac{G^2}{2\pi i}A\,,$$

the two terms (5) and (7) cancel.

[3] DII, section V.
[4] M, equation (37).
[5] M, equation (26).

Let us now consider the contribution from the second term of $N(x_2 - x_1)$. Introducing it in (1) one gets

$$(9) \quad -\frac{B}{2}\frac{G^2}{(4\pi)^4}\left\{\int_{-\infty}^{\sigma} dx_1 \int_{-\infty}^{\sigma_1} dx_2\, S_{FF}(x_1 - x')\left[\left(\gamma^\mu \frac{\partial}{\partial x_2} - m\right)\delta(x_1 - x_2)\right]\mathcal{A}_{\sigma_2}(x_2, x'') + \right.$$

$$\left. + \int_{-\infty}^{\sigma} dx_2 \int_{-\infty}^{\sigma_2} dx_1\, S_{FF}(x_1 - x')\left[\left(\gamma^\mu \frac{\partial}{\partial x_2} - m\right)\delta(x_1 - x_2)\right]\mathcal{A}_{\sigma_1}(x_2, x'')\right\}.$$

We show in Appendix I that the two integrals cancel each other and the whole expression vanishes.

We are left therefore with the convergent kernel $N_P^c(x_2 - x_1)$. The renormalization of the lowest order T.D. integral equation is achieved. The correct integral equation is obtained by adding to the r.h.s. of equation (I.11) the following terms:

$$(10) \quad \frac{G^2}{8}\left\{\int_{-\infty}^{\sigma} dx_1 \int_{-\infty}^{\sigma_1} dx_2 [S_{FF}(x_1 - x')N_P^c(x_2 - x_1)\mathcal{A}_{\sigma_2}(x_2, x'') + \right.$$

$$+ S_{FN}(x_1 - x'')N_N^c(x_2 - x_1)\mathcal{A}_{\sigma_2}(x', x_2)] +$$

$$+ \int_{-\infty}^{\sigma} dx_2 \int_{-\infty}^{\sigma_2} dx_1 [S_{FF}(x_1 - x')N_P^c(x_2 - x_1)\mathcal{A}_{\sigma_1}(x_2, x'') +$$

$$\left. + S_{FN}(x_1 - x'')N_N^c(x_2 - x_1)\mathcal{A}_{\sigma_1}(x', x_2)]\right\}.$$

At this point one must emphasize the difference between the present treatment of the bound state problem and the perturbation solution of the collision problems with the S matrix. In the latter case no contribution arises to the matrix element for the scattering, from processes, like the one described by the kernel of equation (1) in which no quanta are exchanged between the two particles: $S^c(p)$ vanishes when p is a free particle four-momentum. But in the present case the contribution (10) should not vanish for two reasons. The first one is that $\mathcal{A}_{\sigma_1}(x_2, x'')$ is not only the product of free particle operators but contains the time dependence of $\Psi(\sigma_1)$; the second one is that the integration on the times t_1 and t_2 is not extended to $+\infty$. Therefore the argument of $S^c(p)$ in $N^c(x_2 - x_1)$ does not satisfy $p^2 = -m^2$ and $S^c(p)$ is different from zero. The separation of $S^c(p)$ from the divergent part is of course quite unambiguous and is justified by the requirement that the same divergent constants are to be used for mass and charge renormalization in the bound state and in the collision problems.

3. – The renormalization of exchange kernels containing primitive divergent graphs.

We call exchange kernels those ones describing the exchange between proton and neutron of at least one meson: we will examine the G^4 divergent kernels of this type. They all contain a primitive divergent graph ([3]) and have been explicitly written in equations (I.25), (I.27), (I.31) and (I.32) ([6]). We will discuss them separately.

3·1. – *The vertex part.* – We rewrite the kernel (I.25) as

$$\mathcal{V} = S_{FP}(x_1 - x')\Lambda_P(x_1 - x_2,\ x_2 - x_3)\Delta_F(x_2 - x_4)S_{FN}(x_4 - x'')\gamma_5^N . \tag{11}$$

The factor

$$\Lambda_P(x_1 - x_2,\ x_2 - x_3) = \gamma_5^P S_{FP}(x_2 - x_1)\gamma_5^P S_{FP}(x_3 - x_2)\gamma_5^P \Delta_F(x_1 - x_3) , \tag{12}$$

is the lowest order irreducible vertex part ([7]). We use the same procedure as for the self energy term (2) and obtain

$$\Lambda_P(x_1 - x_2,\ x_2 - x_3) = 8L\gamma_5^P\,\delta(x_1 - x_2)(x_2 - x_3) + \Lambda_P^c(x_1 - x_2,\ x_2 - x_3) , \tag{13}$$

Here again L is an infinite constant and Λ_P^c a convergent function. L and the Fourier transform of Λ_P^c are defined by MATTHEWS ([8]).

We can now calculate the contribution of kernel (11) to the G^4 integral equation (I.22). We have ([9])

$$^V\mathcal{A}_\sigma(x', x'') = LG^2\,{}^E\mathcal{A}_\sigma(x', x'') + {}^V\mathcal{A}_\sigma^c(x', x'') , \tag{14}$$

where

$$\begin{aligned} {}^E\mathcal{A}_\sigma(x', x'') = \\ = \frac{G^2}{8}\Big\{ \int_{-\infty}^{\sigma} dx_1 \int_{-\infty}^{\sigma_1} dx_2\, S_{FP}(x_1 - x')\gamma_5^P S_{FN}(x_2 - x'')\gamma_5^N \Delta_F(x_1 - x_2)\mathcal{A}_{\sigma_2}(x_1, x_2) + \\ + \int_{-\infty}^{\sigma} dx_2 \int_{-\infty}^{\sigma_2} dx_1\, S_{FP}(x_1 - x')\gamma_5^P S_{FN}(x_2 - x'')\gamma_5^N \Delta_F(x_1 - x_2)\mathcal{A}_{\sigma_1}(x_1, x_2)\Big\} . \end{aligned} \tag{15}$$

([6]) We consider only, for brevity, the radiative processes for the proton.

([7]) A priori one should write Λ_P as a function of x_1, x_2, x_3, but the Fourier analysis shows that actually it is only a function of the difference $x_1 - x_2$ and $x_2 - x_3$.

([8]) M, equation (36).

([9]) The suffix I means that we consider only the contribution of the vertex part. The suffix c indicates the convergent part separated from a divergent expression.

is the exchange term (I.14) of the G^2 equation and

$$(16)\qquad {}^{r}\mathcal{A}^{c}_{\sigma}(x', x'') = \frac{G^4}{64}\Big\{\int_{-\infty}^{\sigma} dx_1 \int_{\sigma_1}^{\sigma} dx_2 \int_{\sigma_1}^{\sigma} dx_3 \int_{\sigma_1}^{\sigma} dx_4\, \mathcal{V}^{c}\mathcal{A}_{\sigma_1}(x_3, x_4) +$$

$$+ \int_{-\infty}^{\sigma} dx_2 \int_{\sigma_2}^{\sigma} dx_1 \int_{\sigma_2}^{\sigma} dx_3 \int_{\sigma_2}^{\sigma} dx_4\, \mathcal{V}^{c}\mathcal{A}_{\sigma_2}(x_3, x_4) + \int_{-\infty}^{\sigma} dx_3 \int_{\sigma_3}^{\sigma} dx_1 \int_{\sigma_3}^{\sigma} dx_2 \int_{\sigma_3}^{\sigma} dx_4\, \mathcal{V}^{c}\mathcal{A}_{\sigma_3}(x_3, x_4) +$$

$$+ \int_{-\infty}^{\sigma} dx_4 \int_{\sigma_4}^{\sigma} dx_1 \int_{\sigma_4}^{\sigma} dx_2 \int_{\sigma_4}^{\sigma} dx_3\, \mathcal{V}^{c}\mathcal{A}_{\sigma_4}(x_3, x_4)\Big\}.$$

In equation (16) $\mathcal{V}^{c}$ is obtained from $\mathcal{V}$ by replacing Λ_P with Λ^{c}_{P}.

3·2. *The proton self energy part.* – The exchange kernels of this type are (I.31) and (I.32), which we rewrite as follows, using equation (2)

$$(17)\qquad \mathcal{S}_1 = S_{FP}(x_1 - x')N_P(x_2 - x_1)S_{FP}(x_3 - x_2)\gamma_5^P S_{FN}(x_4 - x'')\gamma_5^N \Delta_F(x_3 - x_4),$$

$$(18)\qquad \mathcal{S}_2 = S_{FP}(x_1 - x')\gamma_5^P S_{FP}(x_2 - x_1)N_P(x_3 - x_2)S_{FN}(x_4 - x'')\gamma_5^N \Delta_F(x_1 - x_4).$$

Consider the expression of $N_P(x_2 - x_1)$ given in (5). The contributions to equation (I.22) deriving from the infinite constant A in both (17) and (18) cancel, just as it happened for the term (5), with the terms arising from the proton mass renormalization δm. For the contributions containing the infinite constant B we use the same procedure outlined in Appendix I. Summing the results for both kernels (17) and (18) one obtains finally

$$(19)\qquad {}^{s}\mathcal{A}_{\sigma}(x', x'') = G^2 \frac{B}{2\pi i}\, {}^{E}\mathcal{A}_{\sigma}(x', x'') + {}^{s}\mathcal{A}^{c}_{\sigma}(x', x''),$$

${}^{s}\mathcal{A}^{c}_{\sigma}$ is given by an equation identical to equation (16) in which $\mathcal{V}^{c}$ is replaced by

$$(20)\qquad \mathcal{S}^{c} = \mathcal{S}^{c}_1 + \mathcal{S}^{c}_2 = S_{FP}(x_1 - x')[N^{c}_{P}(x_2 - x_1)S_{FP}(x_3 - x_2)\gamma_5^P S_{FN}(x_4 - x'')\gamma_5^N \cdot$$

$$\cdot \Delta_F(x_3 - x_4) + \gamma_5^P S_{FP}(x_2 - x_1)N^{c}_{P}(x_3 - x_2)S_{FN}(x_4 - x'')\gamma_5^N \Delta_F(x_1 - x_4)].$$

3·3. *The meson self energy part.* – The kernel (I.27) is written

$$(21)\qquad \mathcal{M} = - S_{FP}(x_3 - x')\gamma_5^P S_{FN}(x_4 - x'')\gamma_5^N \Delta_F(x_3 - x_1)M(x_1 - x_2)\Delta_F(x_2 - x_4)$$

and the Fourier analysis of $M(x_1 - x_2)$ gives

$$(22) \quad M(x_1 - x_2) = -\frac{2}{\pi} A' \delta(x_1 - x_2) + \frac{2}{\pi} C \left(\frac{\partial^2}{\partial x_1^2} - \mu^2 \right) \delta(x_1 - x_2) + M^c(x_1 - x_2) .$$

The divergent constants A', C, and the convergent function $(p^2 + \mu^2)\Delta^c(p)$, whose Fourier transform is $M^c(x)$ are defined by MATTHEWS [10].

As it happened previously with the nucleon self energy term, the contribution from (22) containing A' cancels with the term arising from the meson mass renormalization $\delta\mu^2$ added to the Hamiltonian, provided we choose

$$(23) \quad \delta\mu^2 = -\frac{G^2}{\pi i} A' .$$

In Appendix II the calculation of the contribution from the C term is performed and discussed: one obtains

$$(24) \quad {}^M\mathcal{H}_\sigma(x', x'') = G^2 \frac{C}{2\pi i} {}^E\mathcal{H}_\sigma(x', x'') + {}^M\mathcal{H}^c_\sigma(x', x'') .$$

Here again ${}^M\mathcal{H}^c_\sigma(x', x'')$ is obtained by replacing $\mathcal{V}^c$ in equation (16) with

$$(25) \quad \mathcal{W}^c = -S_{FP}(x_3 - x')\gamma_5^P S_{FN}(x_4 - x'')\gamma_5^N \Delta_F(x_3 - x_1)\Delta_F(x_2 - x_4) M^c(x_1 - x_2) .$$

It is now apparent that the divergent multiples of ${}^E\mathcal{H}_\sigma(x', x'')$ appearing in equations (14), (19) and (24) can be absorbed in the renormalization of G. Actually one should distinguish between the proton-meson coupling constant (say G^P) and the neutron meson coupling constant (G^N). Terms (14) and (19) contribute to the renormalization of G^P and we label L_P, B_P the divergent constants involved. If G_1^λ is the renormalized coupling constant for either neutron or proton, we write, following Dyson,

$$(26) \quad G_1^\lambda = (Z_{1\lambda})^{-1} Z_{2\lambda} (Z_3)^{1/2} G^\lambda, \qquad \lambda = N, P,$$

where, up to terms of order $(G^\lambda)^2$

$$(27) \quad \begin{cases} Z_{1\lambda} = 1 - (G^\lambda)^2 L , \\ Z_{2\lambda} = 1 + \dfrac{(G^\lambda)^2}{2\pi i} B_\lambda , \\ Z_3 = 1 + \dfrac{1}{2\pi i} [(G^P)^2 C_P + (G^N)^2 C_N] . \end{cases}$$

[10] M, equation (38).

These factors are exactly the same one gets in the S matrix to the same order in the coupling constant.

4. – Renormalization of other kernels.

Other kernels containing reducible (11) graphs can be renormalized with a straightforward use of Dyson's work. Let us consider as an example the self energy kernel (I.28) which we rewrite

$$\mathcal{R} = S'_{FP}(x_1 - x')\gamma_5^P S_{FP}(x_2 - x_1)\, \Lambda_F(x_1 - x_3)\, \Lambda_P(x_3 - x_2, x_4 - x_3)\,. \tag{28}$$

From equation (13) Λ_P can be separated in a divergent part containing L and a convergent Λ_P^c. Because of the δ-functions in the divergent part, one obtains, when $\mathcal{R}$ is introduced in the integral equation (I.22), L times the term (1). The remaining part can be written

$$\mathcal{R}' = S_{FP}(x_1 - x')\gamma_5^P \iiint \exp[-ip(x_2 - x_1)] \exp[-iq(x_3 - x_2)] \cdot \exp[-is(x_4 - x_3]\, \mathrm{d}p\, \mathrm{d}q\, \mathrm{d}s \int R(p, q, s; k)\, \mathrm{d}k\,, \tag{29}$$

with

$$R(p, q, s; k) = \frac{i\gamma^P(p + k) - m}{(p + k)^2 + m^2} \frac{1}{k^2 + \mu^2} \Lambda_P^c(q + k, s)\,. \tag{30}$$

Now the last integral on k in (29) is still divergent, but, according to Dyson, the function Λ_P^c has a sufficiency of denominators for convergence, and therefore it will be at most linearly divergent. Applying then the usual renormalization procedure we can write:

$$\begin{aligned}\mathcal{R}' = S_{FP}(x_1 - x')\gamma_5^P \Big\{ & A''\, \delta(x_1 - x_2)\, \delta(x_2 - x_3)\, \delta(x_3 - x_4) + \\ & + B' \left[\left(\gamma^P \frac{\partial}{\partial x_2} - m\right) \delta(x_1 - x_2)\right] \delta(x_2 - x_3)\, \delta(x_3 - x_4) + \\ & + B''\, \delta(x_1 - x_2) \left[\left(\gamma^P \frac{\partial}{\partial x_3} - m\right) \delta(x_2 - x_3)\right] \delta(x_3 - x_4) + \\ & + B'''\, \delta(x_1 - x_2)\, \delta(x_2 - x_3) \left[\left(\gamma^P \frac{\partial}{\partial x_4} - m\right) \delta(x_3 - x_4)\right] + \mathcal{R}^c(x_1 - x_2, x_2 - x_3, x_3 - x_4)\Big\}.\end{aligned} \tag{31}$$

(11) We use here the word reducible in the sense defined by Dyson (DII, section IV).

where, A'', B', B'', B''' are divergent constants and $\mathcal{R}^c$ is a convergent function. The contribution from A'' cancels with the fourth order proton mass renormalization, and the B terms vanish exactly as it happened with the term (9).

In this way a convergent part can be separated from all the G^4 kernels listed in section 4·2 of I. The integral equation (I.22) is therefore completely divergence-free.

Higher order terms in G contain also the so called « square parts » ([12]). The elimination of this kind of divergences presents no difficulty at all: the third term of (6) added to the Hamiltonian removes the infinite contribution and a finite part is separated and kept.

A similar procedure can be applied to all the higher order divergent kernels, eliminating gradually the divergences starting with the primitive divergent graphs by using the methods of the S matrix: no divergences higher than quadratic arise so that the methods outlined in Appendix I and II are sufficient for their complete removal. We are left therefore — at least in principle — with finite kernels in our integral equation up to any order in the coupling constant.

5. – The disconnected loops in the vacuum.

In equation (I.20) a divergent contribution to the G^2 equation has been found whose kernel describes a closed disconnected loop in the vacuum. Such a contribution, and the other ones of higher order which in the S matrix theory are consistently neglected because they give rise to constant multiplicative phase factors in the probability amplitude with no physical meaning, should, in principle, be kept now because the unknown function $\mathcal{A}_\sigma(x', x'')$ is involved in the integration. On the other hand one notices that only the time dependence of $\Psi(\sigma)$ appears as a variable of integration and it seems justified to adopt for these terms an assumption similar to the perturbation one, namely that the time dependence of $\Psi(\sigma_2)$ can be neglected in the integrand. In the S matrix theory one replaces $\Psi(\sigma_2)$ with $\Psi(-\infty)$: if this is done here $\mathcal{A}_{\sigma_2}(x', x'')$ vanishes for the reasons stated in I. We can alternatively replace $\mathcal{A}_{\sigma_2}(x', x'')$ with $\mathcal{A}_\sigma(x', x'')$ and take it out of the integral which becomes the vacuum expectation value of the operator $U(\sigma, \quad \infty)$ ([13]). This infinite factor in the probability amplitude has no physical significance and can be dropped in this case too.

([12]) M, equation (35).

([13]) This matrix U is the same defined in DI, equation (13).

But one would expect that examining more closely the contribution of these terms it should be possible to extract from them some finite parts in the spirit of the renormalization procedure. Unfortunately it turns out that this is not the case. Let us write the term (I.20) in the form

$$(32) \qquad {}^{I}\mathcal{A}_{\sigma}(x', x'') = -\frac{G^2}{8}\int_{-\infty}^{\sigma} \mathrm{d}x_1 \int_{-\infty}^{\sigma_1} \mathrm{d}x_2\, \mathcal{C}\mathcal{A}_{\sigma_2}(x', x'')\,,$$

with

$$(33) \qquad \mathcal{C} = M(x_1 - x_2)\,\Delta_F(x_1 - x_2)\,.$$

By means of the usual procedure of Fourier analysis one can write

$$(34) \qquad \mathcal{C} = A'''\,\delta(x_1 - x_2) + C''\,\frac{\partial^2}{\partial x_2^2}\,\delta(x_1 - x_2) + \mathcal{C}_c(x_1 - x_2)\,,$$

with A''' and C'' infinite constant and $\mathcal{C}_c$ a finite function. Now the contribution of the term containing A''' in (32) could be cancelled by adding to the Hamiltonian a constant « vacuum renormalization counter-term », but the contribution containing C'' does not vanish, if we use for the calculation the same prescription adopted in Appendix II for the evaluation of the meson self energy graph, namely to extend the integration on x_2 from $-\infty$ to $\sigma_1 + \varepsilon$ [14].

The impossibility of using a renormalization procedure for these processes is rather unsatisfactory: it underlines the fact that when we abandon the hypothesis of perturbation theory new difficulties arise in the quantum theory of fields.

6. – Discussion.

The present method enables to write an integral equation for the probability amplitude $\mathcal{A}_{\sigma}(x', x'')$ completely free from divergences. If one assumes that the renormalized masses and coupling constants are given formally by the same series of infinite constant terms in powers of G_1^2 as in the S matrix

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[14] Of course it vanishes if we exclude the singular point x_1 by performing the integration from $-\infty$ to $\sigma_1 - \varepsilon$, but this is not consistent with our previous assumption.

theory, as it should be if any physical meaning is attached to the renormalization procedure, then finite contributions arise also from self energy-kernels without exchange of quanta between the two particles, like (10), which, until now, have always been dropped in practical calculations: the problem of finding out their actual magnitude is being studied.

There is no need of pointing out that the method outlined here is simply a covariant formulation of the Tamm Dancoff Lévy non adiabatic treatment, and is therefore essentially a weak coupling method. It has been used recently for the study of pion nucleon scattering ([15]) and seems to work remarkably well.

The formalism used looks more cumbersome than the B.S. treatment because of the integrations limited in time and the additional σ dependence of the unknown function. On the other hand it must be remembered that the calculations are always performed assuming flat surfaces $t = \text{const.}$, and that the Fourier component of $\mathcal{A}_\sigma(x', x'')$ (indicated in eq. (A.14) with $\mathcal{F}(j, k)$) actually contains a factor $\delta(k^2 + m^2)\delta(j^2 + m^2)$ which simplifies considerably the calculations: the example has been given of the reduction of equation (I.11) to the form (I.17).

It is again emphasized that, once the equation for $\mathcal{A}_\sigma(x', x'')$ is solved, the calculation of observables other than the energy of the bound state becomes possible.

Whether this method differs substantially from the B.S. treatment is not clear: besides the differences outlined above we recall that, according to the results of I, the kernels contributing to the integral equations are quite different.

The contributions of disconnected loops in the vacuum do not give any more trouble here than in the S matrix theory or in the B.S. formulation of the bound system, under the same assumptions, namely that the time dependence of the state vector can be neglected in the integrals involving these processes. Difficulties arise if one wants to go beyond this approximation.

In conclusion the author wishes to thank Professor Wataghin for his interest and encouragement; Professor B. Ferretti and Dr. B. F. Touschek for a very helpful discussion, and Dr. N. Kemmer for criticism in the early stage of the work.

([15]) S. Fubini: *Nuovo Cimento*, **10**, 564 (1953). Also F. J. Dyson and others in Cornell have studied the same problem, but their results are not yet available to us.

Appendix I.

The first integral in (9) becomes

$$(A.1)\qquad \int_{-\infty}^{\sigma} \mathrm{d}x_1 \int_{-\infty}^{\sigma_1} \mathrm{d}x_2\, S_F(x_1 - x') \left[\left(\gamma \frac{\partial}{\partial x_2} - m\right) \delta(x_1 - x_2)\right] \mathcal{A}_{\sigma_2}(x_2, x'') =$$

$$= \int_{-\infty}^{\sigma} \mathrm{d}x_1\, S_F(x_1 - x')\gamma_\mu \mathcal{A}_{\sigma_1}(x_1, x'') \int_{-\infty}^{\sigma_1} \mathrm{d}x_2 \frac{\partial}{\partial x_{2\mu}} \delta(x_1 - x_2) -$$

$$- \int_{-\infty}^{\sigma} \mathrm{d}x_1\, S_F(x_1 - x') \int_{-\infty}^{\sigma_1} \mathrm{d}x_2\, \delta(x_1 - x_2)\gamma \frac{\partial}{\partial x_2} \mathcal{A}_{\sigma_2}(x_1, x'')\,,$$

by making use of the properties of the δ function

$$(A.2)\qquad \frac{\partial}{\partial x_{2\mu}} [S_F(x_1 - x')\gamma_\mu \mathcal{A}_{\sigma_2}(x_2, x'')\, \delta(x_1 - x_2)] =$$

$$= S_F(x_1 - x')\gamma_\mu \mathcal{A}_{\sigma_1}(x_1, x'') \frac{\partial}{\partial x_{2\mu}} \delta(x_1 - x_2) =$$

$$= S_F(x_1 - x')\left[\gamma \frac{\partial}{\partial x_2} \delta(x_1 - x_2)\right] \mathcal{A}_{\sigma_2}(x_2, x'') + S_F(x_1 - x')\, \delta(x_1 - x_2)\left[\gamma \frac{\partial}{\partial x_2} \mathcal{A}_{\sigma_2}(x_2, x'')\right]$$

and noticing that

$$(A.3)\qquad \delta(x_1 - x_2)\left(\gamma \frac{\partial}{\partial x_2} + m\right) \mathcal{A}_{\sigma_2}(x_2, x'') =$$

$$= \delta(x_1 - x_2)\left[\gamma \frac{\partial}{\partial x_2} \mathcal{A}_{\sigma_2}(x_1, x'') + \left(\gamma \frac{\partial}{\partial x_2} + m\right) \mathcal{A}_{\sigma_1}(x_2, x'')\right],$$

$$(A.4)\qquad \left(\gamma \frac{\partial}{\partial x_2} + m\right) \mathcal{A}_{\sigma_1}(x_2, x'') = 0\,.$$

Secondly we write

$$(A.5)\qquad \int_{-\infty}^{\sigma} \mathrm{d}x_2 \int_{-\infty}^{\sigma_2} \mathrm{d}x_1\, S_F(x_1 - x') \left[\left(\gamma \frac{\partial}{\partial x_2} - m\right) \delta(x_1 - x_2)\right] \mathcal{A}_{\sigma_1}(x_2, x'') =$$

$$= - \int_{-\infty}^{\sigma} \mathrm{d}x_2\, S_F(x_2 - x')\gamma_\mu \mathcal{A}_{\sigma_2}(x_2, x'') \int_{-\infty}^{\sigma_2} \mathrm{d}x_1 \frac{\partial}{\partial x_{1\mu}} \delta(x_1 - x_2) +$$

$$+ \int_{-\infty}^{\sigma} \mathrm{d}x_2\, S_F(x_2 - x') \int_{-\infty}^{\sigma_2} \mathrm{d}x_1\, \delta(x_1 - x_2)\gamma \frac{\partial}{\partial x_1} \mathcal{A}_{\sigma_1}(x_2, x'')\,,$$

because

$$\left(\gamma\frac{\partial}{\partial x_2}+m\right)\delta(x_1-x_2)=\left(-\gamma\frac{\partial}{\partial x_1}+m\right)\delta(x_1-x_2)\,, \tag{A.6}$$

and

$$\frac{\partial}{\partial x_{1\mu}}S_F(x_1-x')\gamma_\mu-mS_F(x_1-x')=-2i\,\delta(x_1-x')\,, \tag{A.7}$$

which gives no contribution to the integral (9) when, as it is necessary for the validity of equation (I.14), x' is later than σ.

Interchanging x_1 and x_2 it is easy to see that (A.1) and (A.5) cancel each other, proving that (9) vanishes.

Appendix II.

We have to calculate the following contribution

$$\begin{aligned}\mathcal{J}=-\frac{G^4}{32\pi}\,C\Big\{&\int_{-\infty}^{\sigma}\mathrm{d}x_1\int_{\sigma_1}^{\sigma}\mathrm{d}x_2\int_{\sigma_1}^{\sigma}\mathrm{d}x_3\int_{\sigma_1}^{\sigma}\mathrm{d}x_4\,X\mathcal{A}_{\sigma_1}(x_3,x_4)+\\
&+\int_{-\infty}^{\sigma}\mathrm{d}x_2\int_{\sigma_2}^{\sigma}\mathrm{d}x_1\int_{\sigma_2}^{\sigma}\mathrm{d}x_3\int_{\sigma_2}^{\sigma}\mathrm{d}x_4\,X\mathcal{A}_{\sigma_2}(x_3,x_4)+\int_{-\infty}^{\sigma}\mathrm{d}x_3\int_{\sigma_3}^{\sigma}\mathrm{d}x_1\int_{\sigma_3}^{\sigma}\mathrm{d}x_2\int_{\sigma_3}^{\sigma}\mathrm{d}x_4\,X\mathcal{A}_{\sigma_3}(x_3,x_4)+\\
&+\int_{-\infty}^{\sigma}\mathrm{d}x_4\int_{\sigma_4}^{\sigma}\mathrm{d}x_1\int_{\sigma_4}^{\sigma}\mathrm{d}x_2\int_{\sigma_4}^{\sigma}\mathrm{d}x_3\,X\mathcal{A}_{\sigma_4}(x_3,x_4)\Big\}\,,\end{aligned} \tag{A.8}$$

where

$$\begin{aligned}X=YZ\qquad &Y=S_{FP}(x_3-x')\gamma_5^P S_{FN}(x_4-x'')\gamma_5^N\,,\\
&Z=\Delta_F(x_3-x_1)\,\Delta_F(x_2-x_4)\left[\left(\frac{\partial^2}{\partial x_1^2}-\mu^2\right)\delta(x_1-x_2)\right].\end{aligned} \tag{A.9}$$

Splitting the ranges of integration in a different way one can write (A.8) as follows

$$\mathcal{J}=-\frac{G^4}{32\pi}\,C\left\{\int_{-\infty}^{\sigma}\mathrm{d}x_4\int_{\sigma_1}^{\sigma}\mathrm{d}x_3\,YJ_4+\int_{-\infty}^{\sigma}\mathrm{d}x_4\int_{-\infty}^{\sigma_1}\mathrm{d}x_3\,YJ_3\right\}, \tag{A.10}$$

with

$$(A.11)\quad J_4 = \int_{\sigma_4}^{\sigma} dx_1 \int_{\sigma_4}^{\sigma} dx_2\, Z\mathscr{A}_{\sigma_4}(x_3, x_4) + \\ + \int_{-\infty}^{\sigma_4} dx_1 \int_{\sigma_1}^{\sigma} dx_2\, Z\mathscr{A}_{\sigma_1}(x_3, x_4) + \int_{-\infty}^{\sigma_4} dx_2 \int_{\sigma_2}^{\sigma} dx_1\, Z\mathscr{A}_{\sigma_2}(x_3, x_4)$$

and J_3 is obtained from J_4 by replacing σ_4 with σ_3 ([16]): only J_4 will be explicitly calculated because the procedure for J_3 is identical. Let us consider the first term of J_4:

$$(A.12)\quad \mathrm{I} = \mathscr{A}_{\sigma_4}(x_3, x_4) \int_{\sigma_4}^{\sigma} dx_1 \int_{\sigma_1}^{\sigma} dx_2\, \Delta_F(x_3 - x_1)\, \Delta_F(x_2 - x_4) \left[\left(\frac{\hat{c}}{\hat{c} x_1^2} - \mu^2 \right) \delta(x_1 - x_2) \right] .$$

Assume σ and σ_4 to be flat surfaces $t = \text{const.}$ and consider the integral on x_2 for a given value of x_1. Such an integral is mathematically defined only when the singularity of the δ function's derivative lies inside ([17]) the interval of integration. This happens for all the values of x_1 except when t_1 reaches the values t or t_4: in this case the expression is not defined and a suitable prescription must be formulated to give a meaning to it. It is clear that the only reasonable definition is obtained by extending the limits of the integration on t_2 from $t_4 - \varepsilon$ to $t + \varepsilon$ so that now the singularity lies always within the interval of integration. Taking the limit for $\varepsilon \to +0$ (but never reaching the value zero) one obtains the correct value of (A.12). This prescription is analogous to the one introduced in the calculation of integrals involving δ_+ functions — namely the shifting of the poles from the real axis of an infinitesimal amount — but the difference is that the latter derives from the choice between several possibilities dictated by physical reasons (elimination of ingoing waves), while the former is the only consistent way of defining a mathematically meaningless quantity.

Similarly the second and third term of J_4 will be defined as

$$(A.13)\quad \begin{cases} \mathrm{II} = \displaystyle\int_{-\infty}^{\sigma_1} dx_1 \int_{\sigma_1 - \varepsilon}^{\sigma} dx_2\, Z\mathscr{A}_{\sigma_1}(x_3, x_4) , \\ \mathrm{III} = \displaystyle\int_{-\infty}^{\sigma_1} dx_2 \int_{\sigma_2 - \varepsilon}^{\sigma} dx_1\, Z\mathscr{A}_{\sigma_2}(x_3, x_4) . \end{cases}$$

The explicit calculation is performed by using the Fourier representation of the Δ_F and δ functions, as well as of $\mathscr{A}_\sigma(x_3, x_4)$ which, according to its de-

([16]) But not x_4 with x_3 in $\mathscr{A}_\sigma(x_3; x_4)$.

([17]) Or outside, but we are not interested in this case here.

finition (I.7) can be written, with the help of (I.3) (I.10) as follows

$$\mathcal{A}_0(x_3, x_4) = \iint \mathrm{d}j\,\mathrm{d}k\,\mathcal{F}(j,k)\exp[ijx_3]\exp[ikx_4]\exp[iEt]\,, \tag{A.14}$$

where

$$E = E_j + E_k - W - i\eta\,, \tag{A.15}$$

has, according to our previous conventions, a small negative imaginary part [18]. Performing the integration on the space variables one obtains

$$\begin{aligned} J_4 = \frac{1}{2^4\pi^6}\iint \mathrm{d}j\,\mathrm{d}k\,\mathcal{F}(j,k)\exp[ijx_3]\exp[ikx_4]\int \mathrm{d}^3\boldsymbol{p}\exp[-i\boldsymbol{p}(\boldsymbol{r}_3-\boldsymbol{r}_4)]\cdot \\ \cdot\iiint \mathrm{d}p_0\,\mathrm{d}k_0\,\mathrm{d}q_0\,\frac{E_p^2-q_0^2}{(E_p^2-p_0^2)(E_p^2-k_0^2)}\exp[ip_0t_3]\exp[-ik_0t_4]\cdot \\ \cdot\Big\{\exp[iEt_4]\int_{t_4}^{t}\mathrm{d}t_1\int_{t_4-\varepsilon}^{t+\varepsilon}\mathrm{d}t_2\exp[-it_1(p_0-q_0)]\exp[-it_2(q_0-k_0)]+ \\ +\int_{\infty}^{t_4}\mathrm{d}t_1\int_{t_1-\varepsilon}^{t}\mathrm{d}t_2\exp[-it_1(p_0-q_0-E)]\exp[-it_2(q_0-k_0)]+ \\ +\int_{-\infty}^{t_4}\mathrm{d}t_2\int_{t-\varepsilon}^{t}\mathrm{d}t_1\exp[-it_1(p_0-q_0)]\exp[-it_2(q_0-k_0-E)]\Big\}\,. \end{aligned} \tag{A.16}$$

Dropping terms of the type

$$\int \mathrm{d}q_0\exp[i\varepsilon q_0] = \delta(\varepsilon)\,, \qquad \int q_0\,\mathrm{d}q_0\exp[i\varepsilon q_0] = \delta'(\varepsilon)\,, \tag{A.17}$$

because we never let ε become exactly zero, and making use of the following integrals [19]

$$\begin{cases} P\int\frac{\exp[i\alpha x]}{x-\beta}\,\mathrm{d}x = i\pi\exp[i\alpha\beta]\,, \qquad P\int\frac{\exp[-i\alpha x]}{x-\beta} = -i\pi\exp[-i\alpha\beta] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\alpha,\beta \text{ real};\ \alpha>0), \\ \int\frac{\exp[i\alpha x]}{x-\beta+i\eta}\,\mathrm{d}x = \int\frac{\exp[-i\alpha x]}{x+\beta-i\eta}\,\mathrm{d}x = 0\,, \quad \int\frac{\mathrm{d}x}{x-\beta\pm i\eta} = \pm\pi i\,, \end{cases} \tag{A.18}$$

[18] This is consistent with the assumption that the mass has a small negative imaginary part, required for the evaluation of the poles of the S_F function.

[19] P means principal value.

one obtains readily

$$J_4 = 2i\mathscr{A}_{\sigma_4}(x_3, x_4)\,\Delta_F(x_3 - x_4) \qquad \text{for} \quad t_3 > t_4. \tag{A.19}$$

In a similar way

$$J_3 = 2i\mathscr{A}_{\sigma_3}(x_3, x_4)\,\Delta_F(x_3 - x_4) \qquad \text{for} \quad t_3 < t_4. \tag{A.20}$$

Then, from (A.10)

$$\mathcal{J} = \left(\frac{G^2}{2\pi i}\,C\right)\frac{G^2}{8}\left\{\int_{-\infty}^{\sigma} dx_4 \int_{\sigma_4}^{\sigma} dx_3\, Y\Delta_F(x_3 - x_4)\mathscr{A}_{\sigma_4}(x_3, x_4) + \right. \tag{A.21}$$

$$\left. + \int_{-\infty}^{\sigma} dx_4 \int_{-\infty}^{\sigma_4} dx_3 Y\,\Delta_F(x_3 - x_4)\mathscr{A}_{\sigma_3}(x_3, x_4)\right\},$$

which can be written

$$\mathcal{J} = \left(\frac{G^2}{2\pi i}\,C\right)_E \mathscr{A}_\sigma(x', x''), \tag{A.22}$$

in agreement with (24).

RIASSUNTO (*)

Nella Parte I del presente lavoro è stato mostrato come il trattamento non adiabatico del problema relativistico di due particelle interagenti in un campo mesonico neutro pseudoscalare può essere fatto in forma covariante. Le equazioni integrali che si ottengono contengono noccioli infiniti corrispondenti a processi radiativi. In questa Parte II si applicano i procedimenti di Dyson e Matthews per la rinormalizzazione della matrice S per separare dai noccioli delle equazioni integrali una parte convergente, e si dimostra che i contributi divergenti possono essere assorbiti in una inosservabile rinormalizzazione della massa e della costante di accoppiamento. Se ammettiamo di esprimere le masse e la carica rinormalizzata con la stessa serie formale a termini divergenti costanti, come nella matrice S, come è necessario fare se si vuol conservare un significato fisico al procedimento di rinormalizzazione, si ottengono contributi finiti anche da noccioli di self-energia di particelle non interagenti. I termini derivanti dalla creazione di coppie virtuali nel vuoto che non interagiscono con i nucleoni possono eliminarsi usando la stessa approssimazione della teoria della matrice S, e cioè trascurando in questi integrali la dipendenza temporale del vettore di stato.

(*) *Traduzione a cura della Redazione.*

The Cosmic Ray Intensity Above the Atmosphere Near the Geomagnetic Pole (*) (+).

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(ricevuto il 16 Marzo 1953)

Summary. — With the help of new high altitude intensity measurements of Meredith, Van Allen and Gottlieb with single Geiger counters near the north geomagnetic pole, an assessment of existing knowledge of the low rigidity end of the primary cosmic ray spectrum is presented. The new Iowa rocket experiments fully confirm and substantially extend previous evidence for the marked flattening of the integral primary cosmic ray spectrum below a magnetic rigidity of about $1.5 \cdot 10^9$ volts. In particular, they indicate a complete or nearly complete absence of the major components (H, He) of the primary radiation in the following spectral regions: (*a*) For hydrogen, the magnetic rigidity range $1.2 \cdot 10^9$ volts to $0.18 \cdot 10^9$ volts (kinetic energy range 590 MeV to 18 MeV). (*b*) For helium, the magnetic rigidity range $1.2 \cdot 10^9$ volts to $0.37 \cdot 10^9$ volts (kinetic energy range 700 MeV to 72 MeV). The low end of these ranges is far below that attainable by any means except rocket-borne apparatus. This observed absence of low rigidity primaries is not inconsistent with a solar dipole moment as large as $0.6 \cdot 10^{34}$ gauss-cm^3. But there may be an entirely different physical cause. Due to the low relative abundance of primary nuclei of $Z > 2$, the present data are not of sufficient accuracy to conclusively exclude their presence in the low rigidity region of the spectrum. A crucial test of the solar cut-off hypothesis is available in this connection. Preliminary rocket flights of pulse ionization chambers have been made with the intention of investigating the intensities of low rigidity heavy nuclei.

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1. – Introduction.

This paper contains a brief discussion of several aspects of the low rigidity end of the primary cosmic ray spectrum and, in particular, a summary of some new measurements which have a bearing on this subject.

2. – Some Properties of the Geomagnetic Field.

The earth's magnetic field has been traditionally one of the principal tools for the analysis of the primary cosmic ray spectrum. Like all pure magnetic spectrometers it furnishes an analysis of the incident beam according to magnetic rigidity, i.e. the ratio of momentum-to-charge (1).

The following summary is quoted from a recent article (2): « The over-all effect of the geomagnetic field on charged cosmic ray primaries can be summarized simply and succinctly as follows:

« 1) Since the magnetic force on a moving charged particle is, at every point on its trajectory, perpendicular to its direction of motion ..., the velocity, momentum, energy, and rigidity of the particle are constants of the motion. Only the form of its trajectory is affected by the magnetic field.

« 2) A particle must possess at least a minimum magnetic rigidity $R_0(\lambda, \omega, \varphi, \theta)$ in order to arrive (from a distant source) at a point at the top of the atmosphere of geomagnetic latitude λ and geomagnetic longitude ω in the direction specified by the azimuthal angle φ and the zenith angle θ.

« 3) The unidirectional spectral intensity of all particles having rigidity greater than R_0 is unaffected by the geomagnetic field.

« 4) To the approximation adopted in this paper, R_0 is independent of geomagnetic longitude ω.

« 5) As a function of geomagnetic latitude λ, R_0 at zenith angle zero is the same for positively and negatively charged particles. However, for all

(1) We shall take the momentum-to-charge ratio, or magnetic rigidity, for a particle as the magnitude of $R = pc/Ze$, wherein c is the speed of light, Ze is the charge on the particle, and p is the magnitude of its momentum. The physical dimensions of rigidity are those of electric potential. It is therefore convenient to express rigidities in the cosmic ray region in billions of volts. The abbreviation GV will be used to represent $1 \cdot 10^9$ volts. Magnetic rigidity should not be confused with kinetic energy, total energy, or the product of momentum by the speed of light, all three of which quantities have the physical dimensions of energy and are commonly expressed in billions of electron volts, abbreviated GeV.

(2) J. A. Van Allen: *Physics and Medicine of the Upper Atmosphere* (U. of. New Mexico, 1952) Chapter XIV, p. 239-266. (Ed. by C. S White and O. A. Bonson).

other values of θ, a plot of R_0 versus φ undergoes a « mirror image reflection » in the north-south geomagnetic plane. Thus, for positively charged particles, R_0 is always greater for easterly azimuths than for westerly azimuths; whereas for negatively charged particles the situation is reversed.

« 6) The earth itself acts as a perfect cosmic ray absorber, thus reducing the bi-directional intensity immediately above the earth's atmosphere by a factor of two from its free space value for all particles whose rigidity exceeds R_0.

« 7) The trajectories of uncharged particles such as neutrons and gamma ray photons are not affected by the magnetic field, as is evident from first principles or by noting that their magnetic rigidities are infinitely great, irrespective of their momenta.

« Thus it is seen that the geomagnetic field acts as a huge rigidity spectrometer for all charged cosmic ray primaries. The magnetic rigidity is usually written in the form

$$R = \frac{pc}{Ze},$$

in which pc is the product of the particle momentum by the speed of light and is conveniently expressed in electron volts; e is the elementary electronic charge; and Z is the number of elementary charges carried by the particle. R is usually measured in volts (not electron volts) since it has the physical dimensions of electrical potential.

« If a particle has a known rigidity and a known charge, then its momentum is immediately found .... If, further, its rest mass is known, then its kinetic energy can also be calculated using the standard relativistic relations.

« It is commonly stated that λ, φ, and θ determine the geomagnetic cutoff energy. This is seen to be a correct statement only for a specified type particle. The statement is quite inadequate for a primary beam comprising various type particles. It is the rigidity or momentum/charge ratio, $R_0(\lambda, \varphi, \theta)$, which is the same for all particles. The momentum and energy are derived quantities-being, in general, different for each type of particle ».

For the purpose of analyzing the (uncollimated) cosmic ray beam the earth's magnetic field suffers from several important limitations. First, the upper limit of its range, in principle, is only 60 GV rigidity. But due to complicating atmospheric effects, its practical upper limit is about 15 GV. Secondly, even within the useful range 0-15 GV, $R_0(\lambda, \varphi, \theta)$ is a quite non-linear function of latitude even for $\theta = 0$. At the geomagnetic equator $dR_0/d\lambda = 0$; near 15° geomagnetic latitude $dR_0/d\lambda$ is about 1.7 GV per 10 degrees of latitude; and the entire range of R_0 from 1 GV to zero is dispersed in a quite non-linear way from 59° to 90°. Finally, R_0 is an even more complicated function of latitude for arbitrary values of φ and θ.

The dispersion characteristics of the geomagnetic field can be illustrated by applying an imaginary test spectrum and calculating the resulting « latitude effect ». Let us take a « flat » differential number-rigidity spectrum in the finite range 0-20 GV. More precisely, let

$$\begin{aligned} j(R)\,\mathrm{d}R &= 0.05\,\mathrm{d}R \quad \text{for} \quad 0 \leqslant R \leqslant 20\ \mathrm{GV} \\ &= 0 \qquad\quad \text{for} \quad R \geqslant 20\ \mathrm{GV} \end{aligned} \qquad \text{(Spectrum } A\text{)}$$

wherein $j(R)\mathrm{d}R$ is the unidirectional intensity of primary particles, irrespective of type, whose magnetic rigidities lie in $\mathrm{d}R$ at R and the total unidirectional intensity of particles

$$J = \int_0^{20\ \mathrm{GV}} j(R)\,\mathrm{d}R = 1\,.$$

Then the total downward vertical moving intensity at geomagnetic latitude λ is simply

$$J(\lambda) = \int_{R_0(\lambda,\,\varphi,\,0)}^{20\ \mathrm{GV}} j(R)\,\mathrm{d}R = [1 - 0.05\ R_0(\lambda, \varphi, 0)]\,.$$

$R_0(\lambda, \varphi, 0)$ is obtained from geomagnetic theory ([3]), and has its greatest value of 14.8 GV at $\lambda = 0^\circ$.

The resulting calculated latitude effect $J(\lambda)$ vs. λ is shown in fig. 1 as the curve labeled A.

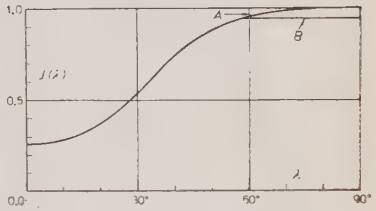

Fig. 1. – Calculated variation of total vertical intensity of primary radiation as a function of latitude for two hypothetical spectra (A) and (B). See text.

For purposes of later discussion let us also consider another simple test spectrum:

$$\begin{aligned} j(R)\,\mathrm{d}R &= 0.05\,\mathrm{d}R \quad \text{for} \quad 1.2 \leqslant R \leqslant 20\ \mathrm{GV} \\ &= 0 \qquad\quad \text{for} \quad 0 \leqslant R \leqslant 1.2\ \mathrm{GV} \qquad \text{(Spectrum } B\text{)} \\ &\qquad\qquad\quad \text{and} \quad R \geqslant 20\ \mathrm{GV} \end{aligned}$$

The calculated latitude effect resulting from this spectrum is labeled (B) in Fig. 1.

Further discussion of Fig. 1 is deferred until a later paragraph.

([3]) R. A. Alpher: *Journ. Geophys. Res.*, **55**, 437-471 (1950).

3. – The Absorption of the Atmosphere.

The principal barrier to the definitive investigation of the low rigidity end of the primary cosmic ray spectrum is the absorption of that portion of the atmosphere which lies above the measuring apparatus. If attention be confined to vertically arriving primaries, then it is a simple matter to list the rigidity (or momentum, or energy) which enables any specified primary nucleus to have an ionization range just equal to a given atmospheric depth h. The value of λ which makes $R_0(\lambda)$, for zenith angle zero, equal to the value of rigidity so listed may be termed the latitude knee «equivalent» to the given atmospheric depth. Otherwise stated, no further experimental latitude effect can be expected for latitudes greater than the latitude which is found to correspond to the given depth, h. A tabulation for representative elements is given in Table I.

TABLE I. – «*Equivalent*» *Latitude Knee.*

Primary Nucleus	Residual Atmospheric Depth, h. 0.5 g/cm²	10 g/cm²	20 g/cm²
H	70°	65°	64°
He	66	60	58
N	64	57	54
Mg	63	55	52
Ca	62	53	50
Fe	61	52	48

It appears from Table I *that it is fruitless with present day balloon techniques to attempt measurements on the proton component of the primary beam at latitudes higher than about* 65°; *on the heavy components, at latitudes higher than about* 52-57°, *depending upon the value of Z.*

The equivalent knees for measurements with omnidirectional equipment such as ionization chambers and photographic emulsions are, effectively, at even lower latitudes than those given in Table I, due to the greater atmospheric depths over a large fraction of the solid angle of arrival of primaries.

The simple considerations above may be subjected to the criticism that, although the primaries themselves may not reach the apparatus, perhaps their presence will be evident at depths greater than their ionization ranges by virtue of their more penetrating secondaries. This criticism has no practical weight in the particle energy region encompassed in Table I so far as μ-meson

secondaries are concerned, nor does it appear to have important weight so far as secondary neutrons are concerned.

4. – Discussion of Figure 1.

As explained above, Fig. 1 shows the expected latitude variation of vertical primary intensity above the atmosphere for two hypothetical primary spectra (*A*) and (*B*), the latter differing from the former only in the complete absence of primaries in the rigidity range $0 \leqslant R \leqslant 1.2$ GV. But it is evident from the preceding section that atmospheric absorption acts in such a way as to convert a « true latitude effect » such as (*A*) into an « experimental latitude effect » resembling (*B*). This example serves to show the great practical difficulty of distinguishing by means of balloon experiments, a genuine absence of low rigidity primaries from an apparent absence due to atmospheric absorption. More precisely, present day balloon techniques offer little or no prospect of learning anything about the primary spectrum at magnetic rigidities less than 0.5 GV.

Such practical limitations on rocket-borne apparatus are much reduced.

5. – Previous Work on the Low Rigidity End of the Primary Spectrum.

In the sea-level measurements of MILLIKAN and NEHER [4] and of COMPTON and TURNER [5], there was an increase of some 10% in the omnidirectional intensity as one passed from the geomagnetic equator to a latitude of 40° (either north or south); and there was no further significant change in progressing to higher latitudes. Thus, 40° was spoken of as the knee in the latitude effect.

But as measurements of various sorts were extended to higher and higher altitudes, the knee was found to move to higher and higher latitudes. An excellent summary of these matters has been given by NEHER [6]. In addition to this summary the most significant investigations dealing with the subject at hand are those of CARMICHAEL and DYMOND [7], of POMERANTZ and

[4] R. A. MILLIKAN and H. V. NEHER: *Phys. Rev.*, **50**. 15 (1936).

[5] A. H. COMPTON and R. N. TURNER: *Phys. Rev.*, **52**, 799 (1937).

[6] H. V. NEHER: *Progress in Cosmic Ray Physics* (Amsterdam, 1952), Chapter V, pp. 243-314. (ed. by J. G. WILSON).

[7] H. CARMICHAEL and E. G. DYMOND: *Proc. Roy. Soc.*, **A 171**, 321 (1939).

McCLURE (8), of PETERSON, NEHER and STERN (9), and of VAN ALLEN and SINGER (10).

The measurements reported in reference (9) comprised a comprehensive series of balloon flights of carefully intercalibrated, steady-drift type ionization chambers. After elimination of temporal fluctuations with the help of comparison flights at a fixed location, it was found that the total rate of ion production in unit vertical column of the atmosphere from sea level to a pressure altitude of 20 g/cm² was unchanged from about 58° to 88° geomagnetic latitude. But as seen from Table I, such measurements do not effectively extend north of 64° latitude. In fact since they are omnidirectional in nature the atmospheric absorption limit is more severe than that used for the basis of Table I, (i.e. vertical incidence). At any rate it appears that these measurements give little or no information on the intensity of primaries of rigidity less than 0.55 GV (kinetic energy of 150 MeV for protons or 40 MeV/nucleon for helium and for other nuclei whose atomic weights are equal to twice their atomic charge). Of course, primary particles can make their presence felt to atmospheric depths greater than their ionization ranges by means of more penetrating secondaries, tertiaries, etc.; in particular, μ-mesons, electrons, and neutrons. The energy range of the bombarding particles in question is within the realm of existing laboratory data on nuclear interactions. From such data it is evident that only neutrons need be considered as significant carriers in this sense. No proper quantitative assessment of this question has yet been made. But it is estimated that the total intensity of primaries of reasonable spectral distribution and of magnetic rigidity less than 0.55 GV could be 10 to 100 times the total intensity of those of rigidity greater than 0.55 GV and could be undetected in the ionization chamber experiments (9).

However, the measurements of Neher and co-workers, by reason of their accuracy and of the corrections for temporal fluctuations, do refine and make more satisfactory certain conclusions (11) (12), previously based upon references (7), (6), and (10). These conclusions derived mainly from the fact that a

(8) M. A. POMERANTZ and G. W. McCLURE: *Phys. Rev.*, **86**, 536 (1952) and references to previous work therein.

(9) V. Z. PETERSON, H. V. NEHER and E. A. STERN: *Phys. Rev.*, **87**, 240 (1952) (A).

(10) J. A. VAN ALLEN and S. F. SINGER: *Phys. Rev.*, **78**, 819 (1950); **80**, 116 (1950).

(11) J. A. VAN ALLEN and S. F. SINGER: *Nature*, **170**, 62 (1952); S. F. SINGER: *Nature*, **170**, 63 (1952).

(12) Note that the energy limit equivalent to the residual atmosphere was stated in reference (7) in terms of electrons, which were at that time erroneously regarded as the most likely primaries. See restatement in terms of protons by H. CARMICHAEL: *Proc. Echo Lake Cosmic Ray Symposium*, June 23-28, 1949, Paper No. 42. (Office of Naval Research, Nov. 1949, Washington, D.C.).

rocket measurement ([10]) of total vertical intensity above the atmosphere at 58° and a balloon measurement ([8]) of total vertical intensity at 69° gave closely the same result. They may be stated in an abridged way as follows:

(*a*) There is a significant absence of primary protons in the spectral range 130-560 MeV kinetic energy.

(*b*) There is a (less) significant absence of primary α-particles in the spectral range 520-670 MeV kinetic energy.

(*c*) No evidence yet exists on the intensities of primary nuclei of Z greater than 2 for rigidities below 1.5 GV. In particular, there is no evidence for the presence or absence of a knee in their spectra.

(*d*) There is no cosmic ray evidence requiring that the general magnetic moment of the sun be less than $0.6 \cdot 10^{34}$ gauss·cm^3. But there is no reason for uniquely ascribing the lack of latitude effect immediately north of 58° to a solar magnetic moment of any such magnitude.

6. – The Need for Further Experiments.

In view of the limitations of present balloon techniques, it appears very difficult, if not practically impossible, to further extend knowledge of the primary spectrum to lower rigidities using balloon borne apparatus. This statement is true of all components but is most sharply true for the heavy nuclear components.

The unknown low-rigidity part of the spectrum is of interest for several reasons:

(*a*) It represents a fundamental part of the incoming radiation.

(*b*) Its nature may serve as a significant limitation on any theoretical proposal for the astrophysical origin and propagation of cosmic rays, and

(*c*) It may yield a significant test of the solar cut-off hypothesis. For example, if all components exhibit spectral knees at the same value of magnetic rigidity, then the solar cut-off hypothesis would remain at least admissible in the sense that it would have survived one definitive test. But if the primary components of various Z values exhibit spectral knees at clearly different values of magnetic rigidity, then an upper limit of the solar moment would necessarily correspond to the lowest observed knee. And in this case, it might well be suggested that the sun's magnetic field had no significant bearing on the observed spectral form.

For these reasons it has seemed worthwhile to conduct further measurements of the low rigidity end of the primary spectrum by using rockets to transport apparatus to much higher altitudes and thus to much smaller atmospheric depths.

7. – The Iowa Experiments.

An inexpensive scheme for transporting some 30 pounds of apparatus to altitudes of at least fifty miles (80 km) has been devised by the Iowa cosmic ray group. The essence of the scheme is to mount the cosmic ray apparatus on a small rocket, lift this entire assembly to an altitude of some fifty thousand. feet with a suitable balloon, then fire the rocket in a near vertical direction. In this way the aerodynamic resistance of the lower atmosphere is eliminated. Altitudes of over fifty miles have been obtained ([13]) at a cost less than that of a large Skyhook balloon flight.

During July-August-September 1952 Lt. M. S. Jones, U.S.N. of the Office of Naval Research led a cosmic ray expedition to the vicinity of the north geomagnetic pole. As part of this expedition, two successful balloon-launched

TABLE II.

Flight No.	S.U.I. No. 4	S.U.I. No. 5
Apparatus	Single Geiger Counter	Single Geiger Counter
Geographical Latitude and Longitude of Release of Balloon	77° 32′ N 73° 29.5′ W	77° 21′ N 73° 29′ W
Date	29 August 1952	29 August 1952
Time of Firing of Rocket	3:30 A.M. E.S.T. (8:30 A.M. G.C.T.)	2:17 P.M. E.S.T. (7:17 P.M. G.C.T.)
Estimated Geomagnetic Latitude of Rocket Flight (Per centered dipole)	89°	89°
Estimated Altitude of Launching of Rocket	36 000 ft above M.S.L.	57 000 ft above M.S.L.
Estimated Summit Altitude	180 000 ft above M.S.L.	295 000 ft above M.S.L.
Duration of Rocket's Flight above 160.000 ft	76 seconds	184 seconds

([13]) J. A. VAN ALLEN and M. B. GOTTLIEB, full paper in preparation for publication elsewhere.

rocket flights of single Geiger counters were made from the U.S.C.G.C. Eastwind near Thule, Greenland. The remainder of this paper is principally concerned with the results of these flights and with their interpretation ([14]). Some pertinent data are assembled in Table II.

Thin-walled (30 mg/cm²) Victoreen 1B85 Geiger counters were contained within ogival aluminum nose cones of about 0.050″ thickness. Only a small fraction of particles from the forward hemisphere of 2π steradians needed to traverse more than 0.5 g/cm² of aluminum before entering the sensitive volume of the counter. A schematic outline of the apparatus is given in Fig. 2. There was a minimum of material in the vicinity of the counter.

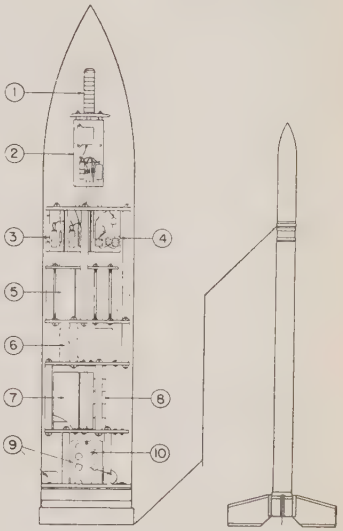

Fig. 2. – Schematic of apparatus as mounted in rocket. Diameter of nose cone 6.5 inches. 1. Single Victoreen 1B85 Geiger counter. 2. Thin fiber plate and cathode follower. 3. Scale-of-32 circuit. 4. Variable frequency audio-oscillator for applying pulses to frequency modulated telemetering transmitter. 5, 6, 7, 8, 9. Batteries. 10. Telemetering transmitter.

The general scheme of the present experiments will be evident by reference to Fig. 3, previously published ([15]). This shows, for a latitude of 41° N, how the counting rate of a single Geiger counter has been found to depend upon altitude. In particular it may be noted that the counting rate is unchanged between about 160 000 ft and the highest altitudes thus far attained. Corresponding data at other latitudes are similar in a general way and exhibit a levelling at about the same altitude. The absolute counting rate, however, has a sharp dependence on latitude and the ratio of the counting rate at the Pfotzer maximum to the counting rate on the high altitude plateau continually diminishes as the latitude increases.

From a knowledge of the « plateau counting rate » and of the dimensions of the effective volume of the counter one can calculate an « average » unidirectional intensity of charged particles. The ideal form of counter for such an experiment is a sphere which presents an equal effective area to all directions

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([14]) L. H. Meredith, J. A. Van Allen and M. B. Gottlieb, full paper in preparation for publication elsewhere.

([15]) A. V. Gangnes, J. F. Jenkins jr. and J. A. Van Allen: *Phys. Rev.*, **75**, 57, 892 (1949).

and thus gives equal weight to every equal element of solid angle. However, the angular distribution of the radiation above the atmosphere is not markedly anisotropic nor is a single counter with a length/diameter ratio of about 3.5 (as used in these measurements) a markedly directional detector. A detailed calculation for the approximately known angular distribution at lower latitudes shows that the $\bar{J}$, computed in the manner described below, differs from the proper average intensity by only two percent when the axis of the counter is vertical and by less than this for other orientations. In previous experiments with the larger length/diameter ratio of 5.7 the counting rate above the atmosphere has varied by less than 5% as the axis of the rocket (and counter) passed through a wide range of zenith angles.

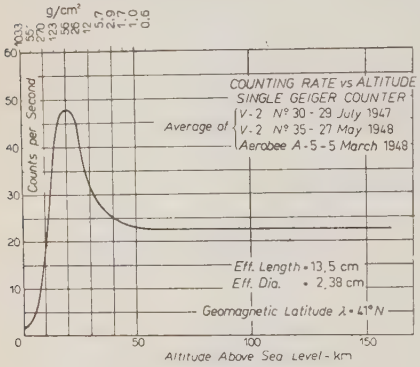

Fig. 3. – Composite experimental curve of counting rate of single Geiger counter as a function of altitude at $\lambda = 41^{\circ}$ N.

With this minor qualification and with the further understanding that any upwards-moving albedo intensity from the lower hemisphere is tacitly assigned to the upper hemisphere, we have referred to $\bar{J} = N/G$ as the average unidirectional intensity over the upper hemisphere. N is the average plateau counting rate and G is the geometrical factor for a single counter for isotropic radiation from one hemisphere only and is given by $G = 0.5\pi^2 al(1 + a/2l)$, where a is the diameter and l is the experimentally measured length of the effective counting volume of the counter. The counters used were also compared by means of a standard radium source-standard distance test in the laboratory. In this test immediately previous to flight, the respective counters of flights No. 4 and No. 5 gave within 0.5% the same counting rates.

It is of special importance to notice that the plateau counting rate is, of course, independent of altitude. Hence, it is not necessary to measure the rocket's altitude, if a sufficiently high altitude is achieved.

The raw counting rate data from our two Arctic flights ([14]) are shown in Figures 4 and 5 respectively. Under the primitive circumstances of these flights it was not feasible to track the rocket. But the Geiger counter itself was used as an altimeter during the balloon portion of the ascension and a rough altitude-time curve for the rocket portion of the flight was calculated from basic data on the performance of the rocket motor, and on the aerodynamic drag function, density of the atmosphere, etc. The time durations of

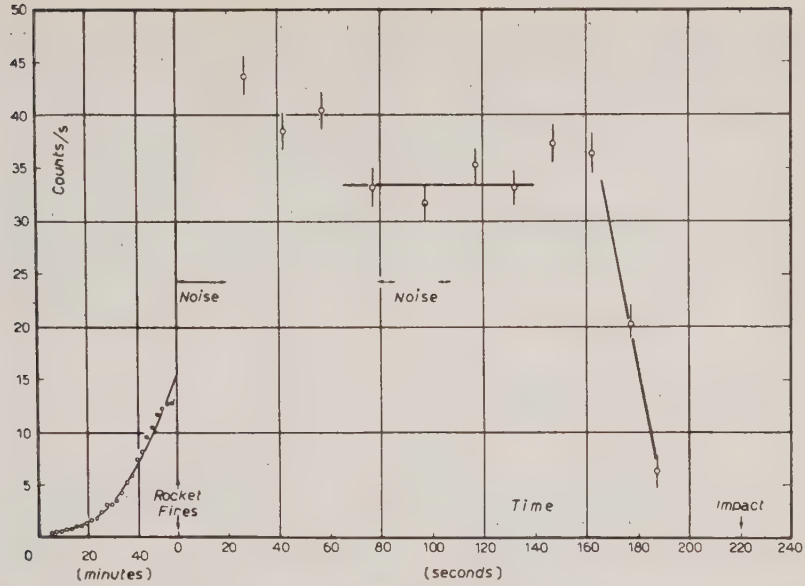

Fig. 4. – Counting rate of single Geiger counter as a function of time of flight. S.U.I. Flight No. 4 near north geomagnetic pole. For the first fifty-four minutes the rocket was being lifted by a General Mills 551 balloon. The remainder of the graph shows data obtained during the rocket's flight. Note sixty-fold expansion of time scale.

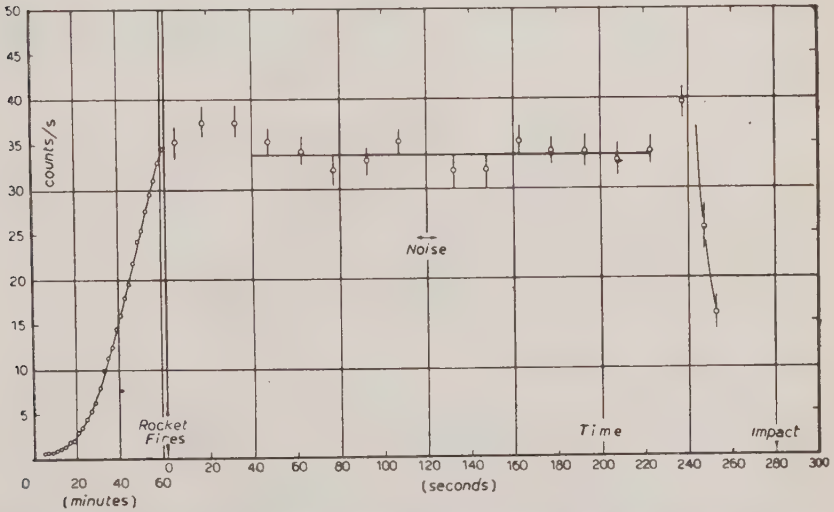

Fig. 5. – Counting rate of single Geiger counter as a function of time of flight. S.U.I. Flight No. 5 near north geomagnetic pole. Rocket fired at 62 minutes after release of rocket-balloon system from ship.

the observed plateaus in the two counting rate vs. time curves are in agreement with such calculations for a near-vertical firing of the rocket and are in disagreement with the firings' having occurred at any large zenith angle. The way in which the r.f. signal strength from the telemetering transmitters varied with time is also in agreement only with near vertical firings.

A counting rate of 39.4 ± 0.1 counts/second was obtained in a long duration balloon flight of identical apparatus, floating for nine hours at about 70 000 feet. This counting rate is seen to be in agreement with the maximum rates observed on the upwards and downwards legs of the trajectories of both flights.

8. – Results and Interpretation.

The average plateau counting rates from flights 4 and 5 are 33.3 ± 0.8 and 33.6 ± 0.5 counts/second, respectively. A total of about 8 000 counts was observed on the two plateaus.

As previously stated in Table II both flights were conducted on 29 August 1952. It is appropriate to inquire whether or not cosmic ray conditions at the times of these firings were typical ([16]) ([17]). Our own data provide no information whatever on this point. But J. A. SIMPSON has kindly shown us unpublished data on the neutron intensity at Climax, Colorado for the period including our firings. These data are of high statistical accuracy and indicate less than 1% deviation from the monthly average during the periods of our flights. Reference ([16]) then provides some confidence that the data obtained in the SUI flights No. 4 and No. 5 are typical of conditions near the geomagnetic pole to within a few percent.

The geomagnetic latitude of the two flights was about 89° N with reference to the centered dipole representation of the earth's magnetic field, as it is usually used by cosmic ray workers. But it is probably more correct to use the eccentric dipole representation ([18]). With respect to it the geomagnetic latitude of our flights was 86° N but the radial distance from the dipole was not significantly different than from a centered dipole. It will be seen below that it is a matter of minor importance, in interpretation of our results, whether the effective latitude was 86° or 89°.

In Table I, it is seen that the absorption limit of the apparatus (about 0.5 g/cm^2 of aluminum) is equivalent for protons to the vertical geomagnetic

([16]) H. V. NEHER and S. E. FORBUSH: *Phys. Rev.*, **87**, 889 (1952).
([17]) J. A. SIMPSON, W. FONGER and L. WILCOX: *Phys. Rev.*, **85**, 366 (1952).
([18]) S. CHAPMAN and J. BARTELS: *Geomagnetism* (Oxford, 1940), Vol. II, p. 644 ff.

cut-off at 70° N. At the summit of flight No. 5 ($\sim$ 295 000 ft) the atmospheric depth ([19]) is less than 0.003 g/cm^2. But this is the only part of the full picture.

At high latitudes and large zenith angles, there is a severe limitation on the magnetic rigidity of incoming primaries due to the earth's shadow cone. This situation can be most readily judeged from the plots of the summary paper of Alpher ([3]). Although the run of these plots appears to some extent intuitively unreasonable, they are direct transcriptions of the basic calculations of Schremp ([20]) and, as such, represent the best available theoretical information on the subject.

It is therefore important to realize that the omnidirectional intensity of primaries above the atmosphere may be expected to continuously increase up to about 89° N *even though* the vertical intensity might show no further increase north of 58° N.

Thus, our new value of $\bar{J}$ is best described as being appropriate to $\lambda = 86°$ N but as including only particles with ranges exceeding 0.5 g/cm^2 of Al. (18 MeV protons, for example). For the 1B85 Geiger counters used ([21]) $l = 6.60 \pm \pm 0.06$ cm, $a = 1.91 \pm 0.02$ cm and hence $G = 71.2 \pm 1.0$ cm^2·steradian and our value of $\bar{J}$ above the atmosphere at $\lambda = 86°$ N is

$$\bar{J} = 0.47 \pm 0.02 \text{ s}^{-1} \text{ cm}^{-2} \text{ steradian}^{-1},$$

with the significance just described. Comparable values of $\bar{J}$ at latitudes 0° and 41° N have been obtained previously ([15]) ([22]). Unfortunately, no strictly

Table III.

λ	$\bar{J}$
0°	0.053 ± 0.002 s^{-1} cm^{-2} steradian^{-1}
41° N	0.131 ± 0.005
50° N	0.31 ± 0.03
58° N	0.41 ± 0.03
86° N	0.47 ± 0.02

([19]) The Rocket Panel. *Phys. Rev.*, **88**. 1027 (1952).

([20]) E. J. Schremp: *Phys. Rev.*, **54**. 158 (1938).

([21]) The reader is cautioned that Meredith has found appreciable differences in the lengths of the central wires and in the experimental effective lengths of various production batches of 1B85's.

([22]) J. A. Van Allen and A. V. Gangnes: *Phys. Rev.*, **78**. 50 (1950).

comparable single counter data exist for latitudes between 41° and 88°. But from counting data for an array of telescopes in previous Aerobee rocket

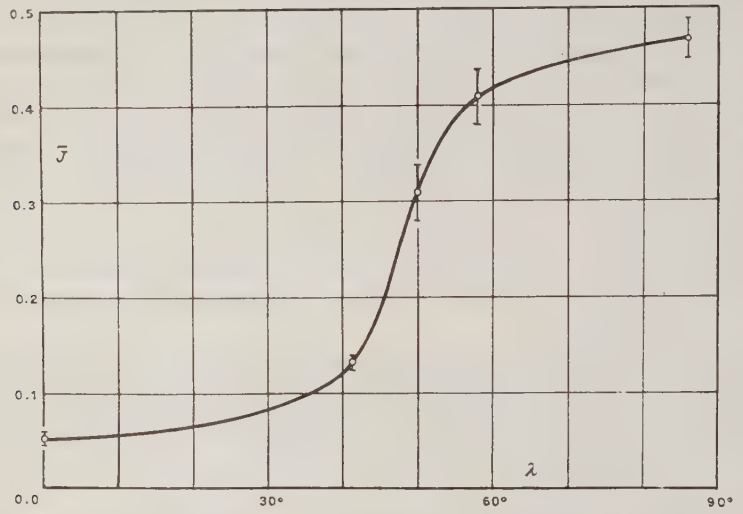

Fig. 6. Experimental values of average unidirectional intensity above the atmosphere at various geomagnetic latitudes. $\bar{J}$ is in absolute units s^{-1} cm^{-2} steradian^{-1}.

flights (1950) to high altitudes at 50° and 58° comparable hemispherical average values of $\bar{J}$ have been synthesized. These data are collected in Table III and are plotted in Fig. 6.

It is important to note that the vertical intensity above the atmosphere at 0°, 41°, 50°, and 58° is found to be less than the intensity at any other zenith angle. Since this is clearly contrary to the expectations of geomagnetic theory for either positively or negatively charged primaries or any mixture of positively and negatively charged primaries, the explanation of this observation must be found in the existence of a comparable intensity of upward, and subsequently downward, moving secondaries from nuclear bursts in the upper atmosphere (albedo). The situation at the geomagnetic equator is shown schematically in Fig. 7. Such a picture seems to also account for the fact that the observed azimuthal asymmetry is much less than that expected from the

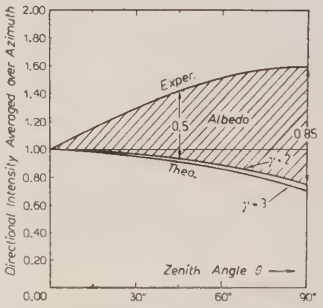

Fig. 7. – Schematic description of the contribution of atmospheric albedo to the total intensity above the atmosphere. Drawn for $\lambda = 0°$.

latitude effect ([23]) ([24]). And it is in approximate accord with simple calculations based on the cross-section for nuclear disintegrations in the atmosphere and on the multiplicity, angular distribution and range of the secondaries so produced.

The vertical intensity above the atmosphere apparently provides the nearest approach to a directly measurable value of the primary intensity, although it doubtless includes albedo radiation to the extent of several percent. The omnidirectional intensity above the atmosphere comprises primary and secondary radiation in comparable amounts.

9. – Conclusions.

Our new value of J at 86° N is grossly inconsistent with an extrapolation to low rigidities of the primary spectrum determined by measurement of the vertical intensity at a series of lower latitudes. Thus, if the integral number-rigidity spectrum of VAN ALLEN and SINGER ([10]) be extrapolated from its proper range of 2.6 GV to 15 GV down to a rigidity of 0.18 GV (corresponding to the value which a proton needs to just penetrate our apparatus) and if a reasonable allowance for albedo be made, the resulting expected value of $\overline{J}$ at 86° N is *more than seven times as great* as that observed.

Or if alternatively, the equivalent integral number-energy spectrum of WINCKLER ([24]) be similarly extrapolated down to 18 MeV, the resulting expected value of $\overline{J}$ at 86° is over twenty times that observed.

Thus it is at once overwhelmingly evident that the empirical integral number-rigidity spectrum of the primary radiation undergoes a marked flattening at magnetic rigidities below about $1.5 \cdot 10^9$ volts.

More particularly, the data of Table III (and Fig. 6) are consistent with a *complete absence of primaries of magnetic rigidity less than* $1.2 \cdot 10^9$ *volts*, since the observed increase of $\overline{J}$ between 58° and 86° is completely accounted for by the additional contribution of higher rigidity particles which are inevitably admitted as the earth's shadow cone opens up as one progresses through this range of latitude ([25]).

([23]) J. A. VAN ALLEN and A. V. GANGNES: *Phys. Rev.*, **79**, 51 (1950).

([24]) J. R. WINCKLER, T. STIX, K. DWIGHT and R. SABIN: *Phys. Rev.*, **79**, 656 (1950).

([25]) The previously mentioned flatness of Neher's ionization data from 58° to 88° (85° by the eccentric dipole) can apparently be understood only on the basis that the additional high energy primaries which are admitted by the opening of the earth's shadow cone come in at sufficiently large zenith angles that they expend their energy in the upper skin of the atmosphere above 20 g/cm².

Thus, the new Iowa rocket experiments fully confirm and substantially extend previous evidence (see Section 5) for the paucity of low rigidity primaries. In particular, they indicate a complete or nearly complete absence of the major components (H, He) of the primary radiation in the following spectral regions:

(*a*) For hydrogen, the magnetic rigidity range $1.2 \cdot 10^9$ volts to $0.18 \cdot 10^9$ volts (kinetic energy range 590 MeV to 18 MeV).

(*b*) For helium, the magnetic rigidity range $1.2 \cdot 10^9$ volts to $0.37 \cdot 10^9$ volts (kinetic energy range 700 MeV to 72 MeV).

The low end of these ranges corresponds to the absorption limit of our apparatus. It is practical to reduce this still farther, but it is already far below that attainable by any means except rocket-borne apparatus.

The stated uncertainty values in Table III have been estimated so as to include both known systematic errors and statistical errors in the sense that the true values of $\bar{J}$ are thought to be within the stated limits to at least a 75% probability. Thus, an intensity in the above specified spectral regions as great as ten percent of the total would be quite unlikely. It is now within reason, using the inexpensive rocket-balloon technique, to reduce this uncertainty considerably by making a series of flights of identical apparatus in the latitude range 55° to 86°.

The observed absence of low rigidity primaries is not inconsistent with a solar dipole moment as large as $0.6 \cdot 10^{34}$ gauss·cm³. But there may be an entirely different physical cause for their absence.

10. – Low Rigidity Heavy Primaries and the Sun's Magnetic Moment.

Due to the low relative abundance of primary nuclei of $Z > 2$ ($\sim 1\%$ of total intensity at 55°), the above described total intensity data are not of sufficient accuracy to significantly exclude the presence of heavy nuclei in the low rigidity region of the spectrum. A technique more specific to the properties of the heavy nuclei is required ([26]). With this objective, two rocket flights of spherical pulse ionization chambers were also made during the 1952 Arctic expedition.

The results are very tantalizing in suggesting a relatively large intensity of low rigidity heavy nuclei. But the specific arrangement of apparatus has not yet received sufficient proof to justify confidence in these preliminary results. This work is being continued since it seems to offer the greatest likelihood of providing definitive evidence on the solar hypothesis for the exclusion of low rigidity particles from the primary beam as it is received at the top of the

([26]) J. A. Van Allen: *Phys. Rev.*, **84**, 791 (1951).

earth's atmosphere; or more generally, of providing evidence on the relationship of heavy components to the lighter components.

Finally, it may be remarked that it already appears that the hydrogen and helium components exhibit about the same magnetic rigidity cut-off in their spectra. This fact favors the solar cut-off hypothesis.

RIASSUNTO (*)

Servendosi dei risultati delle nuove misure d'intensità a grandi altezze eseguite da MEREDITH, VAN ALLEN e GOTTLIEB con singoli contatori Geiger, vicino al polo geomagnetico Nord, si presenta un quadro d'insieme delle conoscenze attuali sull'estremità di bassa rigidità dello spettro primario dei raggi cosmici. I nuovi esperimenti con razzi eseguiti nello Iowa pienamente confermano e sostanzialmente ampliano le prove preesistenti di un appiattimento marcato dello spettro primario integrale dei raggi cosmici al disotto di una rigidità magnetica di circa $1{,}5 \cdot 10^9$ V. In particolare se ne può dedurre l'assenza completa o quasi completa dei componenti principali della radiazione primaria (H, He) nelle seguenti regioni spettrali: (*a*) Per l'idrogeno, l'intervallo di rigidità magnetica da $1{,}2 \cdot 10^9$ V a $0{,}18 \cdot 10^9$ V (intervallo d'energia cinetica: 590 MeV - 18 MeV). (*b*) Per l'elio, l'intervallo di rigidità magnetica da $1{,}2 \cdot 10^9$ V a $0{,}37 \cdot 10^9$ V (intervallo d'energia cinetica: 700 MeV - 72 MeV). L'estremità inferiore di questi intervalli è molto al disotto di quella raggiungibile con qualsi mezzo, eccetto le apparecchiature portate da razzi. L'assenza di primaria di bassa rigidità che è stata constatata non è incompatibile con un momento di dipolo solare fino a $0{,}6 \cdot 10^{34}$ gauss $\cdot$ cm^3; può però esser dovuta ad una causa fisica del tutto differente. A causa dell'abbondanza relativamente scarsa di nuclei primari con $Z > 2$, i dati attualmente disponibili non sono sufficientemente precisi per escludere la loro presenza nella regione dello spettro di bassa rigidità magnetica. A tal riguardo è possibile una verifica cruciale dell'ipotesi del cut-off solare. Sono stati eseguiti con razzi lanci preliminari di camere di ionizzazione a impulso allo scopo di studiare le intensità dei nuclei pesanti di bassa rigidità.

(*) *Traduzione a cura della Redazione.*

On the Hamiltonian structure of non-local field theories. (*)

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(ricevuto il 23 Marzo 1953)

Summary. — For the example of the Kristensen-Möller theory expressions are given for energy, momentum and charge which are constant in the course of time as a consequence of the field equations. For the case that the fields are classical and that the manifold of solutions is the same as for free fields (normal class), canonical field variables fulfilling the Hamiltonian formalism are proved to exist. In the first order approximation of the perturbation theory the canonical field variables are actually constructed also for quantized fields.

Introduction.

By non-local theories we understand in this paper a type of theories (both classical or quantized) in which the field equations (in mechanics the equations of motion) are integro-differential equations containing an arbitrary function F, the so-called form factor, depending on one or more coordinate differences of pairs of points (1). (In the corresponding problem of mechanics this function F depends, of course, only on differences of time values). Therefore, the field equations (equations of motion) of these non-local theories contain explicitly the values of the field quantities (observables) at different times, in contrast to local theories which determine directly the time derivatives of these quantities by their initial values at a given time.

It is the purpose of this paper to clarify the general structure of these theories at least to some extent. First of all the existence of an energy integral in these theories was seriously doubted and only so-called collision invariants

(*) The content of this paper has been communicated at the meeting on Non-Local Field Theories, held in Turin from March 9th to March 14th. (*Editor's Note*).

(1) In connection with the divergence difficulties of quantized lorentzinvariant field theories, such a non-local field theory with a lorentzinvariant form factor was first proposed by G. Wataghin: *Zeits. f. Phys.*, **86**, 92 (1934). Since then and particularly more recently, a large literature on this subject occurred.

have been established, which keep the same value in the limits both for $t \to -\infty$ and for $t \to +\infty$, but which are not constant in between (2). However, in contrast to this view it has been correctly emphasized by A. S. WIGHTMAN (3) that the invariance of a quantized theory with respect to any continuous group must necessarily be connected with the existence of corresponding integrals (constants in time) of the field equations (equations of motion) and that these integrals also determine the variations of the field quantities (observables) of the theory for an infinitesimal transformation of this group (4).

Taking the Kristensen-Möller theory (5) of the interaction of neutral scalar (or pseudoscalar) mesons with nucleons (6) as an example of non-local field theories, we derive first in § 1 of this paper the explicit expressions for the integrals of energy-momentum and of charge in this theory. The usual variational methods are hereby avoided as not convenient in non-local theories and the problem is simply treated by direct calculation. The method can be applied for any non-local theory and also guarantees that the expressions for the integrals obtained in this way go over into the usual expressions for the local theory in the limit where the form factor degenerates into a product of δ-functions. Moreover, the relations postulated by the group theory are necessary conditions for the commutation laws of the field quantities in a non-local theory.

Both in non-local field theories and in non-local mechanical theories (7)

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(2) Compare particularly C. BLOCH: *Det. Kong. Danske Vidensk. Selskab, Mat.-fys. Meddelelser*, **26**, No. 1 (1950) and **27**, No. 8 (1952). (The latter paper is quoted in the following as « C. Bloch II »). From the failure of a particular variation method to lead to the integrals the incorrect conclusion of their non-existence has been too quickly derived.

(3) Discussions at the Copenhagen Physics Conference, June 1952.

(4) To the group of the translations of space-time corresponds in this way the energy-momentum integral W_μ with the commutation law equation (1.13) of this paper, to the group of the multiplication of the spinors ψ and $\overline{\psi}$ with a constant phase factor e^{ix} and e^{-ix} respectively, the charge integral Q with the commutation law (1.14).

(5) P. KRISTENSEN and C. MØLLER: *Det. Kong. Danske Vidensk. Selskab, Mat.-fys. Meddelelser*, **27**, No. 7 (1952). In the following this paper is quoted as « Kr.-M., l. c. ».

(6) This type of interaction avoids certain complications which arise for non-local theories through the gauge group for electro-magnetic interactions.

(7) The most simple example for the latter is obtained in the case of a harmonic oscillator by replacing in the action integral

$$\tfrac{1}{2}\int q^2 \, dt \quad \text{by} \quad \tfrac{1}{2}\iint q(t)F(t-t')q(t') \, dt \, dt' .$$

This example leads to a linear integro-differential equation which can be rigorously solved by a Fourier transformation of $q(t)$.

a difficult initial-value problem arises, namely the question which values of the field quantities can be arbitrarily given at a certain time instant in such a way that they determine past and future of the system uniquely. Closely connected with this question is the other one whether canonical field quantities or observables, defined by the usual canonical commutation relations, exist in non-local field theories. In § 2 and § 3 it is shown that for a certain class of form factors, the « normal class », for which the initial-value problem has the same answer as for free fields, canonical field quantities always exist for classical fields.

For quantized fields this has been proved only in first approximation of perturbation theory where these quantities can actually be constructed (§ 4).

Independently of these still open problems the methods and results of § 2 and § 3 may be useful for the investigations of others. In § 2 methods of analytical mechanics in arbitrary coordinates are developed, the main result of which is the equivalence of the Jacobi-identity for the Poisson-brackets with the vanishing of the cyclic curl of a covariant antisymmetric tensor of rank two (see eq. 2.12). Although this has been known to me for quite a time I have not found it yet in the literature. In § 3 the bilinear Covariant of the (classical) Kristensen-Möller theory has been derived, in a form, however, which explicitly contains the values of the field quantities at different times.

For the final answer to the question whether non-local theories can actually successfully be applied to physics, other properties of these theories will be decisive which are not treated in this paper and which are more closely connected with the Lorentz group. Even if canonical field quantities will turn out to exist they will not have a simple transformation law of local character. The characteristic difference between local and non-local lorentzinvariant field theories is the fact that in the latter *it is not any longer possible to define field quantities (observables) which commute with each other simultaneously in all space-like pairs of points.* Therefore, in any case, non-local theories have to be considered as an enrichment of the known mathematical possibilities for quantized lorentzinvariant field theories.

For their significance in physics, the further discussion of the convergence of the higher approximations of the perturbation methods applied to these theories will be very important. This question does not seem to me yet sufficiently clarified [8]. One will have to find out, whether the postulate of

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[8] C. Bloch (see II, quoted in [2]) generally proved the convergence of all approximations of the perturbation theory for such form factors, the Fourier transforms of which vanish for all space-like momentum vectors. Such form factors will, however, in general lead to a wrong time order of processes (acausality) even for macroscopic istances. Moreover they do not contain the local theory as a limiting case.

convergence (possibly together with some new other postulates) can be used to restrict the possibilities for the choice of the form factor which, at present, seems to be much too arbitrary to have in this general form an actual connection with physics.

1. – The integrals of energy-momentum and of charge in the Kristensen-Möller theory.

From the field equations of this theory

$$(1)\qquad \begin{cases} \left(\gamma_\mu \dfrac{\partial}{\partial x^\mu} + M\right)\psi(x) + g\iint \mathrm{d}x''\mathrm{d}x''' F(x, x'', x''')u(x'')\psi(x''') = 0, \\ -\dfrac{\partial\overline{\psi}(x)}{\partial x^\mu}\gamma_\mu + M\overline{\psi} + g\iint \mathrm{d}x'\mathrm{d}x'' F(x', x'', x)\overline{\psi}(x')u(x'') = 0, \\ -(\Box - m^2)u(x) + g\iint \mathrm{d}x'\mathrm{d}x''' F(x', x, x''')\overline{\psi}(x')\psi(x''') = 0, \end{cases}$$

one derives for the energy-momentum tensor $t^0_{\mu\nu}$ of the free particles

$$t^0_{\mu\nu} = \frac{1}{2}\left\{\overline{\psi}\gamma_\nu \frac{\partial\psi}{\partial x^\mu} - \frac{\partial\overline{\psi}}{\partial x^\mu}\gamma_\nu\psi - \delta_{\mu\nu}\left(\overline{\psi}\gamma_\lambda \frac{\partial\psi}{\partial x^\lambda} - \frac{\partial\overline{\psi}}{\partial x^\lambda}\gamma_\lambda\psi\right)\right\},$$

$$- \delta_{\mu\nu}M\overline{\psi}\psi + \frac{\partial u}{\partial x^\nu}\frac{\partial u}{\partial x^\mu} - \frac{1}{2}\delta_{\mu\nu}\left(\frac{\partial u}{\partial x^\lambda}\frac{\partial u}{\partial x^\lambda} + m^2u^2\right),$$

the « equation of motion » [9]

$$(2)\qquad \frac{\partial t^0_{\mu\nu}}{\partial x^\nu} = g\iiint \mathrm{d}x'\mathrm{d}x''\mathrm{d}x''' F(x', x'', x''')\left\{\delta^{(4)}(x' - x)\frac{\partial\overline{\psi}(x')}{\partial x'^\mu}u(x'')\psi(x''') + \right.$$

$$\left. + \overline{\psi}(x')u(x'')\frac{\partial\psi(x''')}{\partial x'''}\delta^{(4)}(x''' - x) + \overline{\psi}(x')\frac{\partial u}{\partial x''^\mu}\delta^{(4)}(x'' - x)\psi(x''')\right\}.$$

(In order to avoid complications due to the order of factors I assume here for the sake of simplicity the fields to be c-numbers). We are first interested in energy-momentum after integration over the three dimensional volume.

Putting

$$(3a)\qquad iW_\mu = \int t^0_{\mu 4}\,\mathrm{d}^3x\,,$$

$$(3b)\qquad W_\mu = W^0_\mu + W^{\mathrm{int}}_\mu\,,$$

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[9] Kr.-M.: l. c., p. 12, eq. (25).

one has according to (2)

$$\frac{\mathrm{d}W}{\mathrm{d}t}\, W_\mu = 0\,,$$

if

$$(4a)\qquad \frac{\mathrm{d}W_\mu^{\mathrm{int}}}{\mathrm{d}t} = -g\iiint \mathrm{d}x'\mathrm{d}x''\mathrm{d}x'''F(x',x'',x''')\cdot$$

$$\cdot\Big\{\delta(t'-t)\,\frac{\partial\overline{\psi}(x')}{\partial x'^{\mu}}\,u(x'')\psi(x''') + \overline{\psi}(x')u(x'')\,\frac{\partial\psi(x''')}{\partial x'''^{\mu}}\,\delta(t'''-t) +$$

$$+\,\overline{\psi}(x')\,\frac{\partial u(x'')}{\partial x''^{\mu}}\,\delta(t''-t)\psi(x''')\Big\}\,.$$

By partial integration one gets

$$(4b)\qquad \frac{\mathrm{d}W_\mu^{\mathrm{int}}}{\mathrm{d}t} = +g\iiint \mathrm{d}x'\mathrm{d}x''\mathrm{d}x'''\Big\{\frac{\partial}{\partial x'^{\mu}}\big[F\delta(t'-t)\big] + \frac{\partial}{\partial x''^{\mu}}\big[F\delta(t''-t)\big] +$$

$$+\,\frac{\partial}{\partial x'''^{\mu}}\big[F\delta(t'''-t)\big]\Big\}\,\overline{\psi}(x')u(x'')\psi(x''')\,.$$

(The partial integration seems to be permitted, if only the integrals occurring in (1) do exist).

We use now the translational invariance of F from which follows

$$(5)\qquad \frac{\partial F}{\partial x'^{\mu}} + \frac{\partial F}{\partial x''^{\mu}} + \frac{\partial F}{\partial x'''^{\mu}} = 0\,.$$

Eliminating $\partial F/\partial x''^{\mu}$, one obtains for the middle term in (4)

$$\frac{\partial}{\partial x''^{\mu}}\big[F\delta(t''-t)\big] = -\left(\frac{\partial F}{\partial x'^{\mu}} + \frac{\partial F}{\partial x'''^{\mu}}\right)\delta(t''-t) + i\delta_{\mu 4}\,\frac{\partial\,\delta(t-t'')}{\partial t}\cdot F\,,$$

which gives

$$(6)\qquad \frac{\mathrm{d}W_\mu^{\mathrm{int}}}{\mathrm{d}t} = g\iiint \mathrm{d}x'\mathrm{d}x''\mathrm{d}x'''\Big\{\frac{\partial}{\partial x'^{\mu}}\big[F\big(\delta(t'-t)-\delta(t''-t)\big)\big] +$$

$$+\,\frac{\partial}{\partial x'''^{\mu}}\big[F\big(\delta(t'''-t)-\delta(t''-t)\big)\big] + i\delta_{\mu 4}F\cdot\frac{\partial\delta(t-t'')}{\partial t}\Big\}\,\overline{\psi}(x')u(x'')\psi(x''')\,,$$

and, again by partial integration

$$(7)\qquad \frac{\mathrm{d}W_\mu^{\mathrm{int}}}{\mathrm{d}t} = -g\iiint \mathrm{d}x'\mathrm{d}x''\mathrm{d}x'''F(x', x'', x''')\cdot$$

$$\cdot\left\{\frac{\partial\overline{\psi}(x')}{\partial x'^\mu}u(x'')\psi(x''')\cdot[\delta(t'-t)-\delta(t''-t)] + \overline{\psi}(x')u(x'')\frac{\partial\psi(x''')}{\partial x'''^\mu}\cdot\right.$$

$$\left.\cdot[\delta(t'''-t)-\delta(t''-t)] - i\delta_{\mu 4}\frac{\partial\delta(t-t'')}{\partial t}\overline{\psi}(x')u(x'')\psi(x''')\right\}.$$

This can immediately be integrated with respect to the time (using $\varepsilon(t)=+1$ for $t>0$ and -1 for $t<0$) and gives the final result

$$(8)\qquad W_\mu^{\mathrm{int}} = +g\iiint \mathrm{d}x'\mathrm{d}x''\mathrm{d}x'''F(x', x'', x''')\left\{\frac{\partial\overline{\psi}(x')}{\partial x'^\mu}u(x'')\psi(x''')\cdot\right.$$

$$\cdot\frac{1}{2}[\varepsilon(t'-t)-\varepsilon(t''-t)] + \overline{\psi}(x')u(x'')\frac{\partial\psi(x''')}{\partial x'''^\mu}\cdot\frac{1}{2}[\varepsilon(t'''-t)-\varepsilon(t''-t)]+$$

$$\left.+ i\delta_{\mu 4}\delta(t-t'')\overline{\psi}(x')u(x'')\psi(x''')\right\}.$$

The constant of integration is fixed in such a way that in general $W_\mu \to 0$ for $t\to\pm\infty$.

In the local theory for which $F=\delta^4(x'-x'')\,\delta^4(x''-x''')$ the terms with the ε-function do vanish. The sum $W_\mu^0 + W_\mu^{\mathrm{int}}$ is exactly constant in time (not only equal before and after a collision).

The charge integral can be treated in a similar way. According to KRISTENSEN-MÖLLER, p. 11, eq. (21) one has

$$(9)\qquad \frac{\partial}{\partial x^\mu}(i\overline{\psi}\gamma_\mu\psi) = -ig\int\mathrm{d}x'\int\mathrm{d}x''\int\mathrm{d}x'''\ F(x', x'', x''')\overline{\psi}(x')u(x'')\psi(x''')\cdot$$

$$\cdot[\delta^{(4)}(x'-x)-\delta^{(4)}(x'''-x)].$$

Hence, with

$$(10)\qquad Q^0 = \int\overline{\psi}\gamma_4\psi\,\mathrm{d}^3x\,,\qquad Q\equiv Q^0 + Q^{\mathrm{int}} = \mathrm{const}$$

if

$$(11)\qquad \frac{\mathrm{d}Q_{\mathrm{int}}}{\mathrm{d}t} =$$

$$= +ig\int\mathrm{d}x'\int\mathrm{d}x''\int\mathrm{d}x'''F(x', x'', x''')\overline{\psi}(x')u(x'')\psi(x''')[\delta(t'-t)-\delta(t'''-t)],$$

so that we get

$$(12)\qquad Q^{\mathrm{int}} = -\,ig\iiint F(x', x'', x''')\overline{\psi}(x')u(x'')\psi(x''')\cdot \cdot\frac{1}{2}[\varepsilon(t'-t)-\varepsilon(t'''-t)]\,\mathrm{d}x'\,\mathrm{d}x''\,\mathrm{d}x'''.$$

In the local theory Q^{int} vanishes.

The integrals W_μ and Q are of fundamental importance for the commutation-relations, as these integrals also generate the infinitesimal translations and gauge transformations respectively. This is expressed by the formulas [10]

$$(13)\qquad \frac{\partial f}{\partial x^\mu} = -\,i\,[W_\mu, f]\,,$$

$$(14)\qquad [Q, \psi] = -\,\psi\,; \quad [Q, \overline{\psi}] = +\,\overline{\psi}\,; \quad [Q, u] = 0\,,$$

from which also follows

$$(15)\qquad [W_\mu, W_\nu] = [W_\mu, Q] = 0\,.$$

The relations (13), (14) together with the field equations (1) are necessary conditions which the commutations laws for the fields have to fulfill. They are, however, not entirely sufficient to determine these laws uniquely [11]. There are different methods to define these laws. We shall do this in § 3 by a consideration of the limits $t \to +\infty$ and $t \to -\infty$.

Finally we show the possibility of defining quantities $t^{\mathrm{int}}_{\mu\nu}(x)$ and $j^{\mathrm{int}}_{\mu}(x)$ localized in space and time, which fulfill the continuity equations

$$(16)\qquad \frac{\partial}{\partial x^\nu}(t^0_{\mu\nu} + t^{\mathrm{int}}_{\mu\nu}) = 0\,; \quad \frac{\partial}{\partial x^\nu}(i\overline{\psi}\gamma_\nu\psi + j^{\mathrm{int}}_\nu(x)) = 0\,,$$

from which the weaker conditions

$$\frac{\mathrm{d}}{\mathrm{d}t}(W^0_\mu + W^{\mathrm{int}}_\mu) = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\int i\overline{\psi}\gamma_4\psi\,\mathrm{d}^3x + Q^{\mathrm{int}}\right) = 0\,,$$

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[10] It can be shown that the commutation rules (13), (14) are in agreement with the general definitions of the commutators (or anticommutators) of fields recently proposed by R. E. Peierls: *Proc. Roy. Soc.*, A **214**, 143 (1952).

[11] Already for the harmonic oscillator the commutation laws of the observables p, q with the energy (analogous to (13)) together with the equations of motion are insufficient to determine the bracket $[p, q]$. (See E. P. Wigner: *Phys. Rev.*, **77**, 711 (1950)). Moreover, if several fields are present, the number of brackets is in general larger than the number of equations (13).

will follow with

$$iW_{\mu}^{\mathrm{int}} = \int t_{\mu 4}^{\mathrm{int}} \mathrm{d}^3x \; ; \qquad iQ^{\mathrm{int}} = \int j_4^{\mathrm{int}}(x)\mathrm{d}^3x \; .$$

With the « Ansatz »

$$(17) \qquad t_{\mu\nu}^{\mathrm{int}}(x) = -g \iiint E_{\nu}(x'-x,\, x''-x,\, x'''-x)\bar{\psi}(x')u(x'')\psi(x''')\,\mathrm{d}x'\,\mathrm{d}x''\,\mathrm{d}x''' \; ,$$

one obtains for the validity of (16), using (2) and the translational invariance of F, the condition

$$(18) \qquad \left(\frac{\partial}{\partial x'^{\nu}} + \frac{\partial}{\partial x''^{\nu}} + \frac{\partial}{\partial x'''^{\nu}}\right) E_{\mu\nu}(x',\, x'',\, x''') =$$

$$= \frac{\partial}{\partial x'^{\mu}}\left[F\delta^{(4)}(x')\right] + \frac{\partial}{\partial x''^{\mu}}\left[F\delta^{(4)}(x'')\right] + \frac{\partial}{\partial x'''^{\mu}}\left[F\delta^{(4)}(x''')\right],$$

which does not contain the fields any longer and has always solutions.

In an analogous way one assumes for the current $j_{\mu}^{\mathrm{int}}(x)$ the expression

$$(19) \qquad j_{\mu}^{\mathrm{int}}(x) = -ig \iiint E_{\mu}(x'-x,\, x''-x,\, x'''-x)\bar{\psi}(x')u(x'')\psi(x''')\mathrm{d}x'\,\mathrm{d}x''\,\mathrm{d}x''',$$

and obtains from (9) for E_{μ} the condition

$$(20) \qquad \left(\frac{\partial}{\partial x'^{\nu}} + \frac{\partial}{\partial x''^{\nu}} + \frac{\partial}{\partial x'''^{\nu}}\right) E_{\nu}(x',\, x'',\, x''') = F(x',\, x'',\, x''')\left[\delta^{(4)}(x') - \delta^{(4)}(x''')\right],$$

which also has always solutions.

The localization of charge and energy or momentum in space seems less unique than the values of these quantities after integration over the three-dimensional space and need not be fixed as long as the interaction of the considered fields with the electro-magnetic and the gravitational field respectively is not taken into account.

We did not assume here the invariance of the form factor F with respect to the Lorentz group besides its invariance with respect to translations. If this wider invariance holds, one can find in an analogous way the expressions for the six integrals corresponding to the infinitesimal Lorentz-transformations, which are the six components of a skewsymmetric tensor in four-dimensional space-time, the three spatial of which are the angular momentum.

The existence of the integrals here considered, particularly of the energy integral, opens the possibility to discuss the question of the existence of cano-

nical field variables which should fulfill the usual canonical commutation relations, if the fields are taken at the same time instant. In order to show the existence of these canonical fields at least in the classical case, we first report an analogous theorem in the analytical mechanics of systems with a finite number of degrees of freedom.

2. – Analytical mechanics with P.-B.'s ([12]) in general variables. Existence of canonical variables.

We consider here a mechanical system described at a given instant of time by a finite number N of independent variables x^r with $r = 1, ..., N$ (we shall later restrict N to even numbers $N = 2f$). Between any functions A, B, ... of these variables we define P.-B.'s, which have to fulfill the axioms ([13])

$$[A, B] = -[B, A], \tag{2.1}$$

$$[A, c] = 0 \qquad \text{for a constant } c, \tag{2.2}$$

$$\begin{cases} [A_1 + A_2, B] = [A_1, B] + [A_2, B] \\ [A, B_1 + B_2] = [A, B_1] + [A, B_2], \end{cases} \tag{2.3}$$

$$\begin{cases} [A_1 A_2, B] = [A_1, B]A_2 + A_1[A_2, B] \\ [A, B_1 B_2] = [A, B_1]B_2 + B_1[A, B_2], \end{cases} \tag{2.4}$$

$$\big[A, [B, C]\big] + \big[B, [C, A]\big] + \big[C, [A, B]\big] = 0\,. \tag{2.5}$$

If the x^r are c-numbers (that means commutative), which we assume in this section, it follows from the axioms without the last one, that for any two functions A, B of x^r one has

$$i[A, B] = \frac{\partial A}{\partial x^r}\frac{\partial B}{\partial x^s}\mathcal{J}^{rs} = \sum_{r<s}\left(\frac{\partial A}{\partial x^r}\frac{\partial B}{\partial x^s} - \frac{\partial A}{\partial x^s}\frac{\partial B}{\partial x^r}\right)\mathcal{J}^{rs}, \tag{2.6}$$

where

$$\mathcal{J}^{rs} = -\mathcal{J}^{sr} = i[x^r, x^s], \tag{2.7}$$

([12]) We use here P.-B. as abbreviation for Poisson-bracket, which in quantum mechanics is replaced by a commutator multiplied with i. We use here the notation $[f, g]$ for the *P.-B. times* $(-i)$.

([13]) Compare P. A. M. Dirac: *Quantum-Mechanics*, 2nd edition (Oxford, 1935), Chap. V, § 25.

and the usual convention of summation over double indices is accepted. The last axiom (2.5), the famous Jacobi-identity, imposes further restrictions on the $N(N-1)/2$ functions $\mathcal{J}^{rs}(x)$. These restrictions are formulated more easily if we introduce here the simplifying assumption that the *matrix* $\mathcal{J}^{rs}$ *has a reciprocal* $\mathcal{J}_{rs} = -\mathcal{J}_{sr}$ thus satisfying

$$\mathcal{J}_{rt}\mathcal{J}^{ts} = \mathcal{J}^{st}\mathcal{J}_{tr} = \delta_r^s \equiv \begin{cases} 1 & r = s \\ 0 & r \neq s \end{cases}.$$

This is equivalent to the assumption *that the determinant* $|\mathcal{J}^{rs}|$ *is not identically zero.* It is well known that for a skewsymmetric matrix this is only possible if the number N of the variables x^r is even.

One can show that in the more general case of vanishing determinant it is always possible to reduce the number of independent variables.

From (2.8) it follows

$$\frac{\partial \mathcal{J}^{rs}}{\partial x^m} = -\frac{\partial \mathcal{J}_{kl}}{\partial x^m}\mathcal{J}^{kr}\mathcal{J}^{ls}, \tag{2.9}$$

(Differentiate $\mathcal{J}_{kl}\mathcal{J}^{ls} = \delta_k^s$ with respect to x^m and multiply by $\mathcal{J}^{kr}$).

We evaluate now the Jacobi-identity

$$[[x^r, x^s], x^t] + [[x^s, x^t], x^r] + [[x^t, x^r], x^s] = 0. \tag{2.10}$$

From (2.6) it follows using also (2.9)

$$-[[x^r, x^s], x^t] = i[\mathcal{J}^{rs}(x), x^t] = \frac{\partial \mathcal{J}^{rs}}{\partial x^m}\mathcal{J}^{nt} = \frac{\partial \mathcal{J}_{kl}}{\partial x^m}\mathcal{J}^{kr}\mathcal{J}^{ls}\mathcal{J}^{nt}.$$

From this, one gets

$$-\{[[x^r, x^s], x^t] + [[x^s, x^t], x^r] + [[x^t, x^r], x^s]\} = \tag{2.11}$$
$$= \left(\frac{\partial \mathcal{J}_{kl}}{\partial x^m} + \frac{\partial \mathcal{J}_{lm}}{\partial x^k} + \frac{\partial \mathcal{J}_{mk}}{\partial x^l}\right) \cdot \mathcal{J}^{kr}\mathcal{J}^{ls}\mathcal{J}^{mt}.$$

The Jacobi-identity is, therefore, equivalent to the differential equations

$$\frac{\partial \mathcal{J}_{kl}}{\partial x^m} + \frac{\partial \mathcal{J}_{lm}}{\partial x^k} + \frac{\partial \mathcal{J}_{mk}}{\partial x^l} = 0. \tag{2.12}$$

If one passes from the variables x^r to new variables x'^r (which are invertible functions of the old ones), the $\mathcal{J}^{rs}$ and $\mathcal{J}_{rs}$ transform like a contravariant and covariant skewsymmetric tensor respectivaly. Now it is well known, that the left side of (2.12) is the only tensor which can be constructed by differentiation from the skewsymmetric $\mathcal{J}_{rs}$ in a topological space (without metric). Its vanishing is certainly a necessary condition for the existence of such variables x'^r for which $\mathcal{J}'_{rs}$ become constants (independent of the x'^r). But it is well known that this condition is also sufficient [14]: if (2.12) is fulfilled one can always introduce new variables such that $\mathcal{J}_{kl}$ become constants in the new variables. Moreover, if the determinant $|\mathcal{J}_{rs}| \neq 0$, one can always choose the new variables $p_1, q_1; p_2, q_2; \ldots p_f, q_f$ ($N = 2f$) such that $\mathcal{J}_{rs}$ reaches the normal form [15]:

$$\text{(2.13)} \qquad \mathcal{J}^0_{rs} = \begin{pmatrix} 0 & -1 & & & & & & \\ +1 & 0 & & & & & 0 & \\ & & 0 & -1 & & & & \\ & & +1 & 0 & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ & & 0 & & & & 0 & -1 \\ & & & & & & +1 & 0 \end{pmatrix}.$$

The proof of this theorem, which is very closely connected with the so-called Lie-Koenigs theorem of the existence of canonical variables, is usually given in textbooks [16] by reducing it to the theorem of Pfaff on linear differential forms. From (2.12) it follows first that $\mathcal{J}_{kl}$ can be expressed as a curl of a suitably chosen covariant vector-field $\varphi_k(x)$:

$$\text{(2.14)} \qquad \mathcal{J}_{kl} = \frac{\partial \varphi_l}{\partial x^k} - \frac{\partial \varphi_k}{\partial x^l}.$$

The quoted theorem of Pfaff [17] states that the linear differential form

$$\varphi_k(x)\,\mathrm{d}x^k,$$

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[14] The analogous theorem for a symmetric tensor $g_{rs} = g_{sr}$ is the well known theorem, that the vanishing of the Riemann curvature tensor is also sufficient for the possibility to bring g_{rs} to the euclidean normal form.

[15] A proof of the algebraic part of this theorem for constant $\mathcal{J}_{rs}$ is for instance given in H. Weyl, group theory and quantum mechanics, appendix 3.

[16] See for instance E. T. Whittaker: *Analytical Dynamics*, Chap. 10, § 118.

[17] Cf. e.g. E. Goursat: *Leçons sur le problème de Pfaff*.

can always be transformed into the normal form

$$p_1 \mathrm{d}q_1 + p_2 \mathrm{d}q_2 + \ldots + p_f \mathrm{d}q_f + \mathrm{d}q_{f+1} , \tag{2.15}$$

where the last term may or may not be present. In the former case one has $2f \leqslant N$ in the latter $2f + 1 \leqslant N$. In the new variables, the corresponding transformed $\mathcal{J}_{rs}$ has just the normal form (2.13), where possibly a number of zeros could be added in the scheme of the matrix (even or odd as to whether N is even or odd). The latter complication is, however, excluded, if we assume the determinant of the tensor $\mathcal{J}_{rs}$ to be different from zero. Hence the possibility to reach the normal form (2.13) if (2.12) is fulfilled, is proved.

The new variables p_α, q_α ($\alpha = 1, \ldots, f$) are indeed canonical variables, as the P.-B.'s (2.7) in the new variables get, according to (2.13) and $\overset{\circ}{\mathcal{J}}{}^{rs} = -\overset{\circ}{\mathcal{J}}_{rs}$ just the form

$$[q_r, q_s] = [p_r, p_s] = 0 ; \qquad i[p_r, q_s] = -i[q_s, p_r] = \delta_{rs} . \tag{2.16}$$

Returning now to the general variables we postulate that the equation of motion for any function A of the x^r, not explicitly depending on the time must have the form

$$\dot{A} = +i[H, A] , \tag{2.17}$$

where H is the Hamiltonian of the system. This means, particularly according to (2.7)

$$\dot{x}^s = \frac{\partial H}{\partial x^r} \mathcal{J}^{rs} . \tag{2.18}$$

From this particular case the more general formula (2.17) can backwards easily be derived. For the normal form $\overset{\circ}{\mathcal{J}}{}^{rs} = -\overset{\circ}{\mathcal{J}}_{rs}$ one obtains from (2.18) the usual canonical form of the equations of motion.

From (2.18) follows the existence of a *bilinear covariant* constant in time with these equations of motion. Defining indeed the bracket

$$(DA, \delta B) = -(\delta A, DB) \equiv DA \cdot \delta B - \delta A \cdot DB , \tag{2.19}$$

for two independent variations D and δ of the orbit, one can show that

$$\mathcal{J}_{(D,\delta)} \equiv \mathcal{J}_{rs}(Dx^r, \delta x^s) , \tag{2.20}$$

is a constant of the motion:

$$\frac{\mathrm{d}\mathcal{J}_{(D,\delta)}}{\mathrm{d}t} = 0 , \qquad \mathcal{J}_{(D,\delta)} = \text{const.} \tag{2.21}$$

One has indeed, because the operation $\mathrm{d}/\mathrm{d}t$ commutes with the operations D and δ:

$$\frac{\mathrm{d}}{\mathrm{d}t}(Dx^r, \delta x^s) = \left(D\frac{\partial H}{\partial x^t}\mathcal{J}^{tr}, \delta x^s\right) + \left(Dx^r, \delta\frac{\partial H}{\partial x^t}\mathcal{J}^{ts}\right) =$$

$$= \mathcal{J}^{tr}\frac{\partial^2 H}{\partial x^t \partial x^\mu}(Dx^\mu, \delta x^s) + \mathcal{J}^{ts}\frac{\partial^2 H}{\partial x^t \partial x^\mu}(Dx^r, \delta x^\mu) +$$

$$+ \frac{\partial H}{\partial x^t}\left(\frac{\partial \mathcal{J}^{tr}}{\partial x^\mu}(Dx^\mu, \delta x^s) + \frac{\partial \mathcal{J}^{ts}}{\partial x^\mu}(Dx^r, \delta x^\mu)\right).$$

Hence

$$\mathcal{J}_{rs}\frac{\mathrm{d}}{\mathrm{d}t}(Dx^r, \delta x^s) = \frac{\partial^2 H}{\partial x^s \partial x^\mu}(Dx^\mu, \delta x^s) - \frac{\partial^2 H}{\partial x^r \partial x^\mu}(Dx^r, \delta x^\mu) -$$

$$- \frac{\partial H}{\partial x^t}\frac{\partial \mathcal{J}_{rs}}{\partial x^\mu}\left\{\mathcal{J}^{tr}(Dx^\mu, \delta x^s) + \mathcal{J}^{ts}(Dx^r, \delta x^\mu)\right\}.$$

The first two terms vanish because their first factor is symmetric, their second factor antisymmetric in the two summation indices. One eventually obtains

$$\mathcal{J}_{rs}\frac{\mathrm{d}}{\mathrm{d}t}(Dx^r, \delta x^s) = -\frac{\partial H}{\partial x^t}(Dx^r, \delta x^s)\mathcal{J}^{t\mu}\left(\frac{\partial \mathcal{J}_{\mu s}}{\partial x^r} + \frac{\partial \mathcal{J}_{r\mu}}{\partial x^s}\right),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\mathcal{J}_{rs}(Dx^r, \delta x^s)\right\} = -\frac{\partial H}{\partial x^t}(Dx^r, \delta x^s)\mathcal{J}^{tu}\left(\frac{\partial \mathcal{J}_{us}}{\partial x^r} + \frac{\partial \mathcal{J}_{ru}}{\partial x^s} + \frac{\partial \mathcal{J}_{sr}}{\partial x^u}\right),$$

which is, indeed, zero, according to the Jacobi-identity and its consequence (2.12).

For a later application we note that the latter equation is also equivalent to

$$d\mathcal{J}(D, \delta) + D\mathcal{J}(\delta, d) + \delta\mathcal{J}(d, D) = 0\,, \tag{2.12a}$$

for any three commuting variations d, D, δ and $\mathcal{J}(D, \delta)$ given by (2.20), which is another form of the Jacobi-identity. For the canonical variables $\mathcal{J}(D, \delta)$ reduces to the well known form

$$\mathcal{J}(D, \delta) \equiv \sum_{\varrho=1}^{f}(Dq^\varrho, \delta p^\varrho)\,. \tag{2.20a}$$

We can apply these results also to classical field theories if we disregard all convergence questions, which occur with the passing from a finite to an

infinite number of degrees of freedom. This means we simply substitute for the index r the continuous space coordinate $\boldsymbol{x}$ and for the $x^r(t)$ the field $u_\alpha(\boldsymbol{x}, t)$ Instead of $i[x^r, x^s] = \mathcal{J}^{rs}$ one has here the functional relation

$$i\big[u_\alpha(\boldsymbol{x}, t),\, u_\beta(\boldsymbol{x}', t)\big] = \mathcal{J}_{\alpha\beta}(u_\gamma(y);\, \boldsymbol{x}, \boldsymbol{x}')\,.$$

The generalization of the results of this § to this case claims as a consequence of the assumed axioms [18] the existence of canonical fields $p_\alpha(\boldsymbol{x}, t)$, $q_\alpha(\boldsymbol{x}, t)$ which are functionals of the original fields and for which the canonical P.-B.'s hold, namely

$$\big[p_\alpha(\boldsymbol{x}, t),\, p_\beta(\boldsymbol{x}', t)\big] = \big[q_\alpha(\boldsymbol{x}, t),\, q_\alpha(\boldsymbol{x}', t)\big] = 0\,,$$

$$i\big[p_\alpha(x, t),\, q_\beta(\boldsymbol{x},' t)\big] = \delta_{\alpha\beta}\delta^3(\boldsymbol{x} - \boldsymbol{x}')\,.$$

We are now going to apply these theorems to the problem of the existence of canonical field variables in field theories with a form factor (so-called non-local field theories).

3. – The bilinear covariant in classical non-local field theories. The initial-value problem.

We are treating again the Kristensen-Möller theory as an example and start with the variational equations corresponding to the field equations (1.1):

$$(3.1a)\quad \left(\gamma_\mu \frac{\partial}{\partial x^\mu} + M\right)\delta\psi(x) + g\iiint \mathrm{d}x'\,\mathrm{d}x''\,\mathrm{d}x'''\,F(x', x'', x''')\,\delta^{(4)}(x' - x)\,\cdot\,\cdot\,\big(\delta u(x'')\psi(x''') + u(x'')\delta\psi(x''')\big) = 0\,,$$

$$(3.1b)\quad -\frac{\partial\delta\overline{\psi}}{\partial x^\mu}\gamma_\mu + M\,\delta\psi + g\iiint \mathrm{d}x'\,\mathrm{d}x''\,\mathrm{d}x'''\,F(x', x'', x''')\,\delta^{(4)}(x''' - x)\,\cdot\,\cdot\,\big(\delta\overline{\psi}(x')u(x'') + \overline{\psi}(x')\,\delta u(x'')\big) = 0\,,$$

$$(3.1c)\quad -(\Box - m^2)\,\delta u(x) + g\iiint \mathrm{d}x'\,\mathrm{d}x''\,\mathrm{d}x'''\,F(x', x'', x''')\,\delta^{(4)}(x'' - x)\,\cdot\,\cdot\,\big(\delta\overline{\psi}(x')\psi(x''') + \overline{\psi}(x')\cdot\delta\psi(x''')\big) = 0\,.$$

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[18] The Jacobi-identity (eq. (2.10)) is written here

$$\frac{\delta\mathcal{J}_{\alpha\beta}(u_\varrho(y);\boldsymbol{x}', \boldsymbol{x}'')}{\delta u_\gamma(\boldsymbol{x}''')} + \frac{\delta\mathcal{J}_{\beta\gamma}(u_\varrho(y);\, \boldsymbol{x}'', \boldsymbol{x}''')}{\delta u_\alpha(\boldsymbol{x}')} + \frac{\delta\mathcal{J}_{\gamma\alpha}(u_\varrho(y);\, \boldsymbol{x}''', \boldsymbol{x}')}{\delta u_\beta(\boldsymbol{x}'')} = 0\,.$$

Now we multiply these three equations with $D\psi(x)$ from the left, $D\psi(x)$ from the right and Du respectively, add, integrate over the three-dimensional x-space and antisymmetrize the results with respect to two independent variations D and δ. The result is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3x\Big\{(-i)\,(D\overline{\psi}\gamma_4,\,\delta\psi)+\Big(Du,\,\delta\frac{\partial u}{\partial t}\Big)\Big\}+$$
$$+\,g\iiint \mathrm{d}x'\mathrm{d}x''\mathrm{d}x'''\,F(x',x'',x''')\big\{(D\overline{\psi}(x',)\,\delta u(x''))\psi(x''')\cdot$$
$$\cdot\big[\delta(t-t')-\delta(t-t'')\big]+(D\overline{\psi}(x'),\;\delta\psi(x'''))u(x'')\big[\delta(t-t')-\delta(t-t''')\big]+$$
$$+\,(Du(x''),\,\delta\psi(x'''))\overline{\psi}(x')\big[\delta(t-t'')-\delta(t-t''')\big]\big\}=0\,.$$

From this follows the constance in time of the bilinear covariant

$$\mathcal{J}(D,\delta)=\int \mathrm{d}^3x\Big\{-i\,(D\overline{\psi}\gamma_4,\,\delta\psi)+\Big(Du,\,\delta\frac{\partial u}{\partial t}\Big)\Big\}+ \tag{3.2}$$
$$+\,g\iiint \mathrm{d}x'\mathrm{d}x''\mathrm{d}x'''\;F(x',x'',x''')\big\{(D\overline{\psi}(x'),\,\delta u(x''))\psi(x''')\cdot$$
$$\cdot\,\frac{1}{2}\big[\varepsilon(t-t')-\varepsilon(t-t''')\big]+(D\overline{\psi}(x'),\,\delta\psi(x'''))u(x'')\cdot\frac{1}{2}\big[\varepsilon(t-t')-\varepsilon(t-t''')\big]+$$
$$+\,\overline{\psi}(x')(Du(x''),\,\delta\psi(x'''))\cdot\frac{1}{2}\big[\varepsilon(t-t'')-\varepsilon(t-t''')\big]\big\}\,.$$

Moreover, one easily confirms the validity of the Jacobi-identity in the form (2.12a), as the factors of everyone of the three quantities $\varepsilon(t-t')$, $\varepsilon(t-t'')$ and $\varepsilon(t-t''')$ cancel there separately.

The expression (3.2) of the bilinear covariant has, however, not the usual form, as the values of the fields at different times enter it explicitly. These quantities are not independent due to the variational equations (3.1).

At this place, the fundamental problem arises, which quantities can at a given time $t=t_0$ be prescribed arbitrarily in such a way that they determine the future (and past) completely. The answer will in general depend on the choice of the form factor. Both the general results of Pais and Uhlenbeck [19] on exponential form factors (without zeros) and the discussion of particularly simple examples [20] seem to indicate the existence of a wide class

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[19] A. Pais and G. E. Uhlenbeck: *Phys. Rev.*, **79**, 145 (1950).

[20] For the example mentioned in note [7], p. 649, the necessary and sufficient condition for the normal class is that the equation

$$-\nu^2+\omega^2 f(\nu)=0\,,$$

has only two roots. Here ω is a given constant and $f(\nu)$ is the Fourier transform of $F(t-t')$.

of form factors, for which the manifold of solutions of the field equations (1.1) is the same as in the absence of any interaction ($g = 0$). This would mean that for this class of form factors (which we may call the *normal* class) $\psi(x)$ (or $\overline{\psi}(x)$), $u(x)$ and $\partial u(x)/\partial t$ can be given arbitrarily for any instant $t = t_0$ and determine the future (and past) behaviour of the fields completely. Even for this normal class of form factors to which alone we restrict ourselves in the following it is, in general, not possible to give explicit solutions for this initial-value problem in non-local field theories. One can try, however, to find such a solution by power series in the coupling constant g, the convergence of which is then a new problem in itself. The problem of an actual construction of canonical field variables is therefore very intimately connected with the initial-value problem.

The physical meaning of the bilinear covariant (3.2) and of the P.-B.'s which follow from it according to (2.8), (2.20) ([21]) is made more obvious by their connection with the ingoing and outgoing fields defined as the limits of the fields for $t = -\infty$ and $t = +\infty$ respectively. These limits exist for general wave packets with the exception of plane waves. The ingoing and outgoing fields fulfill the field equations for free fields and are connected with the general fields by simple integral equations ([22]). Moreover in these limits the second term (proportional to g) in (3.2) will vanish as a consequence of the diluition of the wave packets, as it consists of products of three fields. Hence we have:

$$\mathcal{J}(D, \delta) = \int \mathrm{d}^3x \left\{ -i(D\overline{\psi}^{\mathrm{in}}\gamma_4, \delta\psi^{\mathrm{in}}) + \left(Du^{\mathrm{in}}, \delta \frac{\partial u^{\mathrm{in}}}{\partial t}\right)\right\} = \tag{3.3}$$

$$= \int \mathrm{d}^3x \left\{ -i(D\overline{\psi}^{\mathrm{out}}\gamma_4, \delta\psi^{\mathrm{out}}) + \left(Du^{\mathrm{out}}, \delta \frac{\partial u^{\mathrm{out}}}{\partial t}\right)\right\}.$$

This equality expresses the theorem of Bloch ([23]) for classical fields that the P.-B.'s for ingoing and outgoing fields are similar to each other and to the usual commutation laws of free fields.

For the normal class the P.-B.'s are now uniquely determined. We can also show the compatibility of the relations (13,) (14) with these P.-B.'s between the fields. By passing to the limits $t = -\infty$ or $t = +\infty$ in these

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([21]) For the normal class the Liouville theorem guarantees the non-vanishing of the determinant $\| J_{rs} \|$.

([22]) C. Bloch II, appendix II and p. 25.

([23]) C. Bloch II, eq. (4.3) and (4.5), pp. 23 and 24. See also eq (4.1) below. I do not consider here possible complications stemming from the existence of negative frequencies in the classical fields. For quantized fields the eigen-field of the particle and its self-energies will give rise here to difficulties which can only be overcome by a change of variables equivalent to a renormalization.

relations one can indeed first replace the fields by the ingoing or outgoing fields and then neglect the interaction part of energy, momentum and charge. Generally one has, therefore.

$$W_\mu(\psi(x), \overline{\psi}(x), u(x)) = W^0_\mu(\psi^{\text{in}}(x), \overline{\psi}^{\text{in}}(x), u^{\text{in}}(x)) = \\ = W^0_\mu(\psi^{\text{out}}(x), \overline{\psi}^{\text{out}}(x), u^{\text{out}}(x)) ,$$

$$Q\ (\psi(x), \overline{\psi}(x), u(x)) = Q^0(\psi^{\text{in}}(x), \overline{\psi}^{\text{in}}(x) = \\ = Q^0(\psi^{\text{out}}(x), \overline{\psi}^{\text{out}}(x) .$$

Hence the relations (13), (14) are equivalent both with

$$(3.4a) \qquad \frac{\partial f^{\text{in}}}{\partial x^\mu} = -i\left[W^0_\mu\ (\text{in}), f^{\text{in}}\right];\ \left[Q^0\ (\text{in}), \overline{\psi}^{\text{in}}\right] = \overline{\psi}^{\text{in}};\ \left[Q^0\ (\text{in}), \psi^{\text{in}}\right] = -\psi^{\text{in}} ,$$

and with

$$(3.4b) \qquad \frac{\partial f^{\text{out}}}{\partial x^\mu} = -i\left[\left[W^0_\mu\ (\text{out}), f^{\text{out}}\right];\ \left[Q^0\ (\text{out}), \overline{\psi}^{\text{out}}\right] = \overline{\psi}^{\text{out}};\right. \\ \left.\left[Q^0\ (\text{out}), \psi^{\text{out}}\right] = -\psi^{\text{out}} .\right.$$

These equations are obviously compatible with the commutation laws for free fields.

4. – Commutation relations and canonical variables in the approximation linear in the coupling constant.

The first approximation of perturbation theory, in other words the terms linear in g, is, however, relatively simple and in this approximation it is easy to find the canonical field variables, also for quantized fields.

For quantized fields it is convenient to use in this approximation the ingoing or outgoing fields and their quantization according to the rules for free fields ([24]). Denoting both kinds of fields with an upper index 0, one has

$$(4.1) \qquad \begin{cases} \psi(x) = \psi^0(x) + g\iiint \left(S \mp \frac{1}{2} S\right)(x - x')F(x', x'', x''')u(x'')\psi(x''')\,\mathrm{d}x'\,\mathrm{d}x''\,\mathrm{d}x''', \\ \overline{\psi}(x) = \overline{\psi}^0(x) + g\iiint F(x', x'', x''')\overline{\psi}(x')u(x'')\left(\overline{S} \pm \frac{1}{2} S\right)(x''' - x)\,\mathrm{d}x'\,\mathrm{d}x''\,\mathrm{d}x''' , \\ u(x) = u^0(x) - g\iiint F(x', x'', x''')\psi(x')\left(\Delta \mp \frac{1}{2}\Delta\right)(x - x'')\psi(x''')\,\mathrm{d}x'\,\mathrm{d}x''\,\mathrm{d}x''' . \end{cases}$$

([24]) Kr.-M., l. c., p. 14, 15.; C. Bloch II, p. 23, 24.

Here the upper (lower) sign corresponds to incoming (outgoing) fields, as can easily be seen by remembering that $S^{\mathrm{ret}} = \overline{S} - \frac{1}{2}S$, $S^{\mathrm{av}} = \overline{S} + \frac{1}{2}S$ and symilar relations for Δ^{ret}, Δ^{av}. The definitions of the functions S, Δ, etc., are the usual ones ([25]) and one has

$$\text{(4.2)} \qquad \begin{cases} \{\psi^0(x), \overline{\psi}^0(x')\} = -iS(x-x') , \\ [u^0(x), u^0(x')] = i\Delta(x-x') . \end{cases}$$

In the first approximation in g one may replace the brackets for the fields $\psi(x)$, $\overline{\psi}(x)$, $u(x)$ by the corresponding brackets of the free fields (index 0). Inserting in this way the values (4.2) of the brackets one obtains, using $\Delta(-x) = -\Delta(x)$, $\overline{\Delta}(x) = -\frac{1}{2}\varepsilon(x)\Delta(x)$, $\overline{S} = -\frac{1}{2}\varepsilon S$:

$$\text{(4.3)} \qquad \begin{cases} [\psi(x), u(y)] = \\ \quad = -\frac{1}{2} ig \iiint S(x-x')F(x', x'', x''')\Delta(x''-y)[\varepsilon(x-x') + \varepsilon(x''-y)] \cdot \\ \qquad \cdot \psi(x''')\mathrm{d}x'\mathrm{d}x''\mathrm{d}x''' , \\ [u(x), \overline{\psi}(y)] = \\ \quad = -\frac{1}{2} ig \iiint F(x', x'', x''')\Delta(x-x'')\overline{\psi}(x')S(x'''-y) \cdot \\ \qquad \cdot [\varepsilon(x'''-y) + \varepsilon(x-x'')]\mathrm{d}x'\mathrm{d}x''\mathrm{d}x''', \\ \{\psi(x), \overline{\psi}(y)\} = -iS(x-y) + \frac{1}{2} ig \iiint S(x-x')F(x', x'', x''')u(x'') \cdot \\ \qquad \cdot S(x'''-y)[\varepsilon(x-x') + \varepsilon(x'''-y)]\mathrm{d}x'\mathrm{d}x''\mathrm{d}x''' , \\ [u(x), u(y)] = i\Delta(x-y) . \end{cases}$$

It is important that the terms without the ε, which spring from the terms with the two possible signs in (4.1), cancel. From this it follows that in the approximation considered the assumption of similar commutation laws for incoming and outgoing fields is selfconsistent.

The form of the commutation laws (4,3) suggests to consider the new fields

$$\text{(4.4}a\text{)} \qquad \varphi(x) = \psi(x) + \frac{1}{2} g \iiint S(x-x')F(x', x'', x''')[\varepsilon(x-x') + \varepsilon(x''-x)] \cdot \\ \cdot u(x'')\psi(x''')\mathrm{d}x'\mathrm{d}x''\mathrm{d}x''' ,$$

$$\text{(4.4}b\text{)} \qquad \overline{\varphi}(x) = \overline{\psi}(x) + \frac{1}{2} g \iiint \overline{\psi}(x')u(x'')F(x', x'', x''')S(x'''-x) \cdot \\ \cdot [\varepsilon(x'''-x) + \varepsilon(x-x'')]\mathrm{d}x'\mathrm{d}x''\mathrm{d}x''' ,$$

([25]) S corresponds to the nucleon mass M, Δ to the meson mass m.

because, according to (4.2) and (4.3) these fields obviously commute with $u(y)$ in the approximation considered:

$$[\varphi(x),\, u(y)] = [u(x),\, \overline{\varphi}(y)] = 0\,.$$

Moreover, it turns out that for two points at equal times the anticommutator of $\varphi(x)$ and $\overline{\varphi}(y)$ is the same as for free fields. One first finds

$$\{\varphi(x),\, \overline{\varphi}(y)\} = -\,iS(x-y) - \frac{i}{2}\,g\iiint S(x-x')F(x',\,x'',\,x''')u(x'')\,\cdot \tag{4.6}$$

$$\cdot\, S(x'''-y)[\varepsilon(x''-x) + \varepsilon(y-x'')]\,\mathrm{d}x'\,\mathrm{d}x''\,\mathrm{d}x'''\,.$$

But for $t_x = t_y$ the two ε's in the last bracket have always opposite signs and cancel. Using $S = i\gamma^4\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')$ for $t_x = t_y$ one finds, therefore,

$$\{\varphi(\boldsymbol{x}, t),\, \overline{\varphi}(\boldsymbol{x}', t)\} = \gamma^4\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')\,, \tag{4.6a}$$

which means that q and its complex conjugate, together with u and $\partial u/\partial t$, are indeed canonical variables in first approximation.

It is interesting to express energy, momentum and charge ((1.3), (1.8) and (1.10), (1.12) [26]) by the new variables $\varphi(x)$, $\overline{\varphi}(x)$. For this purpose one has to use the explicit solutions of the initial-value problem of the free fields given by

$$\psi^0(x) = -\,i\gamma^4\int\limits_{t'=\mathrm{const}} \mathrm{d}^3x'\,S(x-x')\psi_0(x')\,,$$

$$\overline{\psi}^0(x) = \int\limits_{t'=\mathrm{const}} \mathrm{d}^3x'\,\overline{\psi}^0(x')S(x'-x)(-\,i\gamma_4)\,,$$

$$u^0(x) = \int\limits_{t'=\mathrm{const}} \mathrm{d}^3x'\left[\frac{\partial\Delta(x-x')}{\partial t'}\,u^0(x') - \Delta(x-x')\,\frac{\partial u^0(x')}{\partial t'}\right].$$

Using these relations one checks that in the final expressions for W_μ and Q all terms containing ε's cancel (in the approximation linear in g) and one gets

$$W_\mu = -\,i\int \mathrm{d}^3x\left\{\frac{1}{2}\left[\overline{\varphi}\gamma_4\,\frac{\partial\varphi}{\partial x^\mu} - \frac{\partial\overline{\varphi}}{\partial x^\mu}\,\gamma_4\varphi - \delta_{\mu 4}\left(\overline{\varphi}\gamma_\lambda\,\frac{\partial\varphi}{\partial x^\lambda} - \frac{\partial\overline{\varphi}}{\partial x^\lambda}\,\gamma_\lambda\varphi\right)\right] - \right. \tag{4.7}$$

$$\left. -\,\delta_{\mu 4}M\overline{\varphi}\varphi + (-\,i)\,\frac{\partial u}{\partial t}\,\frac{\partial u}{\partial x^\mu} - \frac{1}{2}\,\delta_{\mu 4}\left(\frac{\partial u}{\partial x^\lambda}\,\frac{\partial u}{\partial x^\lambda} + m^2u^2\right)\right\} +$$

$$+\,ig\delta_{\mu 4}\iiint \mathrm{d}x'\,\mathrm{d}x''\,\mathrm{d}x'''\,\delta(t-t'')\overline{\varphi}(x')u(x'')\varphi(x''')F(x',\,x'',\,x''')\,,$$

[26] The formulas hold for q-number fields only apart from the order of factors, but the error is only of higher order in g.

and

$$Q = \int \mathrm{d}^3x\, \overline{\varphi}\gamma_4\varphi\,. \tag{4.8}$$

This cancellation of all additional terms in Q and in W_μ confirms the validity of the commutation laws (1.13) and (1.14) for the new field variables.

The considerations of the last section make it plausible that these results which hold in first approximation in g can be extended to higher orders.

Acknowledgement.

As the considerations presented in this paper have their origin in discussion at the Physics Conference in Copenhagen in June 1952, it is a great pleasure for me to thank Professor NIELS BOHR for having enabled me to participate on this conference. I also thank Prof. R. E. PEIERLS for many interesting discussions.

RIASSUNTO

Nel caso della teoria di Kristensen e Möller, si ottengono delle espressioni per l'energia, il momento e la carica che sono costanti nel tempo come conseguenza delle equazioni dei campi. Nel caso in cui i campi siano classici e l'insieme delle soluzioni sia lo stesso dei campi liberi (classe normale) si dimostra l'esistenza di variabili di campo canoniche che soddisfanno al formalismo hamiltoniano. Anche nel caso di campi quantizzati, nella prima approssimazione della teoria delle perturbazioni, le variabili canoniche dei campi vengono costruite esplicitamente.

Properties of Form Factors in Non-Local Theories (*).

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(ricevuto il 15 aprile 1953)

Summary. — The conditions for absence of non-causal actions over large time intervals in non-local theories are expressed as conditions on the Fourier transform of the form function.

1. – Introduction.

Following the suggestions of Wataghin (1), McManus (2), and others, several authors have recently investigated non-local field equations, i.e. equations which contain integrations over space and time. The purpose of this is to avoid the infinities arising from point interactions, so as to make the use of divergent renormalization factors unnecessary, or at least to interpret them as limits of finite and consistent theories.

In this note we shall not discuss the question of these infinities and the extent to which they are removed by the use of form factors, but only the question of the conditions that must be satisfied to ensure macroscopic causality.

In such non-local theories the Lorentz invariance of the form factors means in general that the factor takes appreciable values near the ligth cone, so that the field equations may link directly fields at very distant points in space-time, if they lie on each other's light cone. There would be no objection to a dependence of events at one point on what happens on its *past* light cone since such actions may be transmitted by a physically possible disturbance

(*) The content of this paper has been communicated at the meeting on Non-Local Field Theories, held in Turin from March 9th to March 14th. (*Editor's Note*).

(1) G. Wataghin: *Zeits. f. Phys.*, **88**, 92 (1934) and **92**, 547 (1935).

(2) H. Mc Manus: *Proc. Roy. Soc.*, A **195**, 323 (1949).

in any case, but a dependence on the field variables at distant points on the *future* light cone would be in conflict with causality.

In equations derived from an action principle the effect of distant past events cannot occur directly in the field equations without that of distant future events occurring in some of the equations as well.

One would not be surprised on the other hand to find in the equations terms referring to slightly later times, the intervals being of the order of the elementary distance r_0 defined by the form factor itself. This distance cannot in any case be longer than about $3 \cdot 10^{-13}$ cm, corresponding to a time of 10^{-23} seconds. The result that the time development of the field could not be described causally over times of this order would not conflict with experience. In fact, within the framework of a non-local theory the interpretation of observations as referring to a time fixed with greater accuracy than r_0/c is bound to meet with difficulty.

We want therefore to find conditions which ensure that the form factors do not lead to terms in the field equations which in reasonable circumstances will depend on events distant in time by much more than r_0/c.

In a recent paper by PAULI (3) the existence of form factors satisfying this condition has been assumed and we shall show here that such functions do, in fact, exist.

This problem has also been studied by C. BLOCH (4). He derives conditions which are sufficient to ensure causality in the sense explained above. We shall find here conditions which are somewhat weaker and therefore allow a wider class of functions than those of BLOCH.

2. – Two-point Form Factors.

We start with the case of a function depending on two points only, as used for example in the equations of MCMANUS (2). If the points are x', x'', translational and Lorentz invariance allow the function $F(x', x'')$ to depend only on

$$s = (\boldsymbol{r}' - \boldsymbol{r}'')^2 - (t' - t'')^2 , \tag{1}$$

where $\boldsymbol{r}'$, $\boldsymbol{r}''$ are the space parts of x', x'' and $c = 1$. For time-like distances, i.e. negative s, the function could have different values according to the sign of $t' - t''$, however, this freedom does not seem particularly useful.

(3) W. PAULI: *Nuovo Cimento*, **10**, 648 (1953).
(4) C. BLOCH: *Dan. Mat. Fys. Medd.*, **27**, no. 8 (1952).

The Fourier transform

$$g(p) = (2\pi)^{-2} \int \mathrm{d}^4x'' \exp\left[-ip\cdot(x'-x'')\right]F(x'-x'') \tag{2}$$

depends only on

$$p^2 = \boldsymbol{p}^2 - p_0^2 . \tag{3}$$

The form factor occurs in the field equations through an integral of the form:

$$\tilde{\Phi}(x') = \int \mathrm{d}^4x'' F(x'-x'')\Phi(x'') , \tag{4}$$

where Φ is a function which may be regarded as known (for example in the process of solution by successive approximations) and which we shall call the source. $\tilde{\Phi}$ is then the « effective source », which will occur in non-local equations where the local equation would have Φ.

In a local theory, where F becomes the δ-function, $\tilde{\Phi}$ and Φ are identical.

The condition of macroscopic causality requires that for a reasonable choice of Φ, large values of $|t'-t''|$ should give negligible contributions to (4). By a reasonable source function Φ we mean a function of the kind that may itself appear in the process of solving the non-local equations. If, for instance, we inserted for Φ a four dimensional δ function, then $\tilde{\Phi}$ will be the form function itself, which certainly does not go to zero near the light cone. But such singular sources will not occur and the highest frequencies contained in the Fourier transform of Φ will be determined in part by the energies of the fields with which we start, and in part by the spread r_0 of the form function itself. If the highest frequency contained in Φ is of the order $1/\tau$, the function can be expanded in a Taylor series over intervals less than τ.

Now consider the effective source (4), choosing for x', for simplicity, the origin. F is appreciable only if the argument (1) is small, or at most of order r_0^2. Hence

$$t''^2 = r''^2 - s \tag{5}$$

is nearly equal to r'', except in the neighbourhood of the origin, where both are small. We introduce s in place of t'' as variable of integration. Then (4) is

$$\tilde{\Phi}(0) = \int \mathrm{d}^3r \, \frac{\mathrm{d}s}{2\sqrt{r^2-s}} \, F(s)\Phi(\boldsymbol{r}, \sqrt{r^2-s}) . \tag{6}$$

Now assume

$$r \gg r_0^2/\tau \qquad \text{and} \qquad r \gg r_0 . \tag{7}$$

Then we may expand Φ and the square root in powers of s:

$$\tilde{\Phi}(0) = \int \frac{\mathrm{d}^3 r}{2r} \left\{ M_0 \Phi(\boldsymbol{r}, r) + M_1 \left[\frac{1}{2r^2} \Phi(\boldsymbol{r}, r) - \frac{1}{2r} \frac{\partial \Phi(\boldsymbol{r}, r)}{\partial t} \right] + \right. \tag{8}$$

$$\left. + M_2 \left[\frac{3}{8r^4} \Phi(\boldsymbol{r}, r) - \frac{3}{8r^3} \frac{\partial \Phi(\boldsymbol{r}, r)}{\partial t} + \frac{1}{8r^2} \frac{\partial^2 \Phi(\boldsymbol{r}, r)}{\partial t^2} \right] + \ldots \right\},$$

where

$$M_n = \int \mathrm{d}s\, s^n F(s). \tag{9}$$

F is of the dimension of an inverse fourth power of a length. Any moment M_n that does not vanish must therefore be expected to be of the order of r_0^{2n-2}. This shows that in the local limit, $r_0 \to 0$, M_0 would tend to infinity, and M_1 to a constant limit. The terms containing these two moments are therefore incompatible with the local limit. M_2 contributes per unit volume a term varying as r^{-3} and the convergence of the space integrals in (8) would therefore be in doubt, unless this also vanishes.

It is therefore necessary that

$$M_n = 0 \qquad \text{for} \qquad n \leqslant N , \tag{10}$$

with N at least 2. A higher value of N may be required if subsequent operations involving the effective source require a decrease faster than the inverse third power of the distance along the light cone.

The non-causal effects decrease faster than any power of the time difference if

$$M_n = 0 , \quad \text{for all } n \tag{11}$$

and this would seem a sufficient condition in any case.

3. – Conditions in Momentum Space.

We now demonstrate how the conditions (10) or (11) may be met. For this purpose consider in (4) the special case of a source Φ which is largest near $t = 0$, and which decreases at positive and negative times in such a way

that the moment

$$K_n = \int \mathrm{d}^4x \, |\Phi(x)|^2 \cdot t^{2r} \tag{12}$$

is finite for any n. This obviously requires that Φ decreases in space faster than r^{-2} and with time faster than any power of $1/t$.

We also assume Φ smooth, and in particular we require also that the integral

$$K_{nl} = \int \mathrm{d}^4x \left| \frac{\partial^l \Phi}{\partial t^l} \right|^2 t^{2n} \tag{13}$$

is finite. These conditions are obviously compatible with those made in deriving (8). An example of a function satisfying (13) is

$$\Phi(x) = \text{const} \cdot \exp\left[-\alpha r^2 - \beta t^2\right]. \tag{14}$$

The moment K_{nl} can be expressed in terms of the Fourier transform of Φ. If

$$\Phi(x) = (2\pi)^{-2} \int \mathrm{d}^4p \, \exp\left[ipx\right] \chi(p), \tag{15}$$

we have

$$K_{nl} = \int \mathrm{d}^4p \left| \frac{\partial^n}{\partial p_0^n} \left(p_0^l \chi(p)\right) \right|^2. \tag{16}$$

Now the effective source is in momentum space

$$\chi(p) = (2\pi)^2 \chi(p) g(p^2), \tag{17}$$

where g is defined in (2).

Hence the moment (12) for the effective source is

$$\begin{aligned} K_n &= \int \mathrm{d}^4x t^{2n} \left| \tilde{\Phi}(x) \right|^2 = \\ &= (2\pi)^4 \int \mathrm{d}^4p \left| \frac{\partial^n}{\partial p_0^n} \left(g(p^2)\chi(p)\right) \right|^2 = \\ &= (2\pi)^4 \int \mathrm{d}^4p \left| \sum \binom{n}{\nu} \frac{\partial^\nu g(p^2)}{\partial p_0^\nu} \frac{\partial^{n-\nu} \chi}{\partial p_0^{n-\nu}} \right|^2. \end{aligned} \tag{18}$$

The ν-th derivative of g with respect to p_0 can be expressed in terms of the first ν derivatives of g with respect to p^2, multiplied by powers of p_0 up to p_0^ν.

Now assume that the first N derivatives of g are bounded

$$\left| \frac{\mathrm{d}^\nu g}{\mathrm{d}(p^2)^\nu} \right| < b_\nu , \qquad \nu \leqslant N . \tag{19}$$

Then one can see that (18) is less than a combination of integrals which are the moments K_{nl} of (16) multiplied by the bounds b_ν for $\nu \leqslant n$. Hence if (19) is valid, $\tilde{K}_n$ is finite for $n \leqslant N$, and therefore $\tilde{\Phi}$ decreases more rapidly than t^{-N}.

Comparison with (8) shows that then the first $N+1$ moments $M_0, \ldots, M_N$ must vanish. Hence (10) is a consequence of (19).

In particular if *all* derivatives of g are bounded, then it follows that (11) holds, which is certainly sufficient.

This is true for the particular function

$$g(p^2) = \frac{(2\pi)^{-2}\lambda^4}{\lambda^4 + (p^2 + m^2)^2} , \tag{20}$$

where λ and m are constants. This function, with $m = 0$, has been used by McMANUS [2].

(11) for this case may also be verified by direct calculation.

This function does not satisfy condition (2.10) of BLOCH [4]. This, of course, is not a contradiction between the two methods, since Bloch's conditions are not claimed to be necessary.

While we have so far proved (19) to be a sufficient condition, it is not much too strong. In particular, if the n-th derivative of g is a step function, the $(n+1)$st derivative contains a δ-function, and therefore it is evident from (18) that the integrand in (18) will contain a δ^2 term and hence $\tilde{K}_{n+1}$ will diverge. This shows that $\tilde{\Phi}(x)$ decreases with time less rapidly than $t^{-(n+3/2)}$. Hence if $n = 0$ so that g itself has a discontinuity, the non-causal effects decrease at most as $t^{-3/2}$, which, according to section 2 shows that $M_0 \neq 0$, which is certainly not admissible.

If the form factor occurs in a theory of the type discussed by McMANUS, where the argument of g denotes the wave vector of a photon, the value of $g(0)$ is fixed by the requirement that the Maxwell equations for long waves be unmodified. In that case the continuity of g implies that $g \neq 0$ both for some space like and for some time like values of p.

4. – Three-point Function.

In theories of the type discussed by RAYSKI [5], KRISTENSEN and MÖLLER [6], and PAULI [3] there occurs a form factor depending on three points, x', x'', x''', which again for reasons of invariance will depend only on the three invariants

$$s' = (x'' - x''')^2 , \qquad s'' = (x''' - x')^2 , \qquad s''' = (x' - x'')^2 . \tag{21}$$

We again introduce a Fourier transform:

$$F(s', s'', s''') = (2\pi)^{-4} \int \mathrm{d}^4p \, \mathrm{d}^4q \exp\left[ip \cdot (x' - x'')\right] \exp\left[iq \cdot (x''' - x'')\right] g(p, q) , \tag{22}$$

where g may be regarded again as a function of three invariants, which may be chosen as

$$p^2 , \qquad (p + q)^2 , \qquad q^2 . \tag{23}$$

Such a function will enter into the field equations through integrals of the type:

$$\tilde{\Phi}(x') = \int \mathrm{d}^4x'' \, \mathrm{d}^4x''' \, F(s', s'', s''') \Phi(x'', x''') , \tag{24}$$

and others obtained by interchanging x', x'', x''', which is equivalent to interchanging the three invariants (23).

We now proceed in momentum space as in section **3** above.

Suppose we use in (24) a source $\Phi(x'', x''')$ which is appreciable only if x'' lies near some point ξ'' and x''' near some point ξ'''. We can express this by writing

$$\Phi(x'', x''') = \psi(x'' - \xi'', \; x''' - \xi''') , \tag{25}$$

where ψ is again a smooth function, appreciable only when all arguments are small, so that in analogy with (13) the moments

$$K_{\mu\nu\varkappa\lambda} = \int \mathrm{d}^4x \, \mathrm{d}^4y \left| \frac{\partial^{\varkappa+\lambda} \psi(x, y)}{\partial x_0^{\varkappa} \partial y_0^{\lambda}} \right|^2 x_0^{2\mu} y_0^{2\nu} , \tag{26}$$

are finite.

We now want to show that the effective source is appreciable only at times t' not very different from τ'', τ''', the time parts of ξ'', ξ'''. For this pur-

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[5] J. RAYSKI: *Phil. Mag.*, **42**, 1289 (1951).

[6] P. KRISTENSEN and C. MÖLLER: *Dan. Mat. Fys. Medd.*, **27**, no. 7 (1952).

pose consider the moment

$$I_m(\xi'', \xi''') = \int \mathrm{d}^4x \, |\tilde{\Phi}(x')|^2 (t' - \tau'')^{2m} . \tag{27}$$

This is also equal to

$$I_m(\xi'', \xi''') = \int \mathrm{d}^4p \left| \left(i \frac{\partial}{\partial p_0} + \tau'' \right)^m \tilde{\chi}(p) \right|^2 , \tag{28}$$

where $\tilde{\chi}(p)$ is the Fourier transform of $\tilde{\Phi}$ which, by using (22) and (24) is also

$$\tilde{\chi}(p) = (2\pi)^2 \int \mathrm{d}^4q \exp[-ip\xi''] \exp[-iq(\xi'' - \xi''')] g(p, q) \chi(p + q, -q) , \tag{29}$$

where $\chi(p)$ is defined by

$$\Phi(x'', x''') = (2\pi)^{-4} \int \mathrm{d}^4p \, \mathrm{d}^4q \exp[ip(x_2 - \xi_2)] \exp[iq(x_3 - \xi_3)] \chi(p, q) . \tag{30}$$

Hence

$$\left(\frac{\partial}{\partial p_0} - i\tau'' \right)^m \tilde{\chi}(p) = \tag{31}$$

$$= (2\pi)^2 \int \mathrm{d}^4q \exp[-ip\xi''] \exp[-iq(\xi'' - \xi''')] \frac{\partial^m}{\partial p_0^m} (g(p, q) \chi(p + q, -q)) .$$

Inserting in (25) and integrating over ξ'' gives

$$\int I_m(\xi'', \xi''') \, \mathrm{d}^4\xi'' = (2\pi)^8 \int \mathrm{d}^4p \, \mathrm{d}^4q \left| \frac{\partial^m}{\partial p_0^m} (g(p, q) \chi(p + q, -q)) \right|^2 . \tag{32}$$

Now g is a function of the invariants p^2, $(p + q)^2$, q^2. Hence we can again write the integrand of (31) as a combination of squares and products of terms of the type

$$\frac{\partial^{\alpha+\beta} g}{\partial (p^2)^\alpha \partial ((p + q)^2)^\beta} \, p_0^\alpha (p_0 + q_0)^\beta \frac{\partial^\mu \chi(p + q, -q)}{\partial p_0^\mu} ,$$

where $\alpha + \beta + \mu \leqslant m$.

Hence if

$$\frac{\partial^{\alpha+\beta} g}{\partial (p^2)^\alpha \partial ((p + q)^2)^\beta} < b_{\alpha\beta} , \qquad \alpha + \beta \leqslant m , \tag{33}$$

(32) can be related with finite coefficients to moments of the type (26), with $\mu + \nu + \varkappa \leqslant m$, $\lambda = 0$. Subject to (33), the moment (32) is therefore bounded and the bound is independent of τ'''. Hence the influence of the source Φ in the effective source $\tilde{\Phi}$ decreases rapidly with the time difference $t' - \tau''$ regardless of whether τ''' is kept constant or allowed to vary with τ''.

Evidently the same considerations apply if we use τ''' instead of τ'', and also if we use the source in the field equations for the functions depending on x'' or x''', which means interchanging x', x'', x''' in (24). A sufficient condition for causality is then that any combined derivatives of g of the form (33) with respect to two of the invariants be bounded.

It follows as before, that a function g which has a discontinuity itself or in a low-order derivative, is inadmissible. Such a function was used by KRISTENSEN and MÖLLER, but is not essential to their reasoning.

One of the authors (M. C.) wishes to express his gratitude to the Swiss Foundation *Pro Helvetia* and the University of Birmingham for financial support.

RIASSUNTO (*)

Le condizioni per l'assenza di azioni non causali in lunghi intervalli di tempo sono espresse nelle teorie non locali come condizioni da imporre alla trasformata di Fourier della funzione di forma.

(*) *Traduzione a cura della Redazione.*

LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

The J Values of States in Configurations $(j)^n$.

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(ricevuto il 28 Febbraio 1953)

The success of the individual-particle model of Meyer and others [1] for the atomic nuclei has revived the study of many problems in theoretical spectroscopy. One problem that arises is the following. For a system of n identical particles of angular momentum j (half integral), a state is given by a set of n distinct magnetic quantum numbers

$$(1) \quad m_1,\ m_2,\ m_3,\ \ldots,\ m_n, \quad -j \leqslant m_i \leqslant j.$$

The total number of states allowed by the Pauli principle is given by

$$(2) \quad k = (2j+1)!/n!(2j+1-n)!.$$

These k states can be grouped into a number of levels of various total angular momentum J whose range of values can be seen to be

$$(3) \quad \left.\begin{matrix} 0 \\ j \end{matrix}\right\} \leqslant J \leqslant J_{\max} = nj - \tfrac{1}{2}\, n(n-1),$$

according as n is even or odd. The problem is to find the allowed values of J for all configurations $(j)^n$ corresponding to all values of j and all $n \leqslant 2j+1$, and in particular, to find the number of levels (each consisting of $2J+1$ states) for each allowed value of J.

No general solution has been given so far for this problem. Extensive tables have been given by Jahn [2] and Flowers [3] for equivalent nucleons in the case of $j \alpha j$ coupling for $j = 3/2,\ 5/2,$ and $7/2$. General formulae have also been deduced by Gamba and Verde [4] in the case of $n = 3$ and 4 for arbitrary j with an extensive use of group theory. From these results it may be seen that an extension to the cases $n > 4$ becomes a very lengthy procedure.

In this note we wish to outline a straightforward method which is applicable for arbitrary j and n without the explicit use of the group theory, although it can be traced to the theory of the permutation group.

To obtain the allowed values of J and the number $\mathcal{N}_J$ of levels of a given J,

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(*) National Research Laboratories Post-doctoral Fellow.

(1) M. Goeppert Mayer: *Phys. Rev.*, **74**, 235 (1948); **75**, 1964 (1949); **78**, 16, 20 (1950).

(2) H. A. Jahn: *Proc. Roy. Soc.*, A **201**, 516 (1950); **205**, 192 (1951).

(3) B. H. Flowers: *Proc. Roy. Soc.*, A **210**, 497 (1952); **212**, 248 (1952).

(4) A. Gamba and M. Verde: *Nuovo Cimento*, **9**, 544 (1952).

let us arrange (1) without loss of generality in the order

$$m_1 > m_2 > m_3 > \ldots > m_n \,, \tag{4}$$

and let us denote the state represented by (1) by the notation

$$(A_1, A_2, A_3, \ldots, A_n), \tag{5}$$

where the A's are positive integers given by

$$\left.\begin{aligned} A_1 &= j - m_1 \,, \\ A_2 &= j - m_2 \,, \\ &\cdots\cdots\cdots \,, \\ A_n &= j - m_n \,. \end{aligned}\right\} \tag{6}$$

It follows from (4) that

$$A_1 < A_2 < A_3 < \ldots < A_n \,. \tag{7}$$

The number N_M of states having a given total z-component M of angular momentum

$$M = m_1 + m_2 + m_3 + \ldots + m_n \tag{8}$$

is given by the number of all possible sets of $m_1, m_2, \ldots, m_n$ subject to (4) and (8). In terms of the A's, the number N_M is given by the number of partitions of the number $nj - M$ in

$$nj - M = A_1 + A_2 + A_3 + \ldots + A_n, \tag{9}$$

subject to (7). The range for M is between $-J_{\max}$ and $J_{\max}$ in (3). But as the number of states with the value M is obviously the same as that of states with $-M$, it is sufficient in the following to consider the positive values of M in the range

$$\left.\begin{matrix} 0 \\ j \end{matrix}\right\} \leqslant M \leqslant J_{\max} \,, \tag{10}$$

according as n is even or odd.

To find N_M, let us for convenience introduce the positive integers $a_1, a_2, \ldots, a_n$ such that

$$\left.\begin{aligned} A_1 &= 0 + a_1 \\ A_2 &= 1 + a_1 + a_2 \\ A_3 &= 2 + a_1 + a_2 + a_3 \\ &\cdots\cdots\cdots\cdots \\ A_n &= (n-1) + a_1 + a_2 + a_3 + \ldots + a_n \,. \end{aligned}\right\} \tag{11}$$

Equ. (9) becomes

$$\begin{aligned} &n a_1 + (n-1)\, a_2 + (n-2) a_3 + \ldots + \\ &+ a_n = nj - \tfrac{1}{2}\, n(n-1) - M = \\ &= J_{\max} - M. \end{aligned} \tag{12}$$

Since the greatest value of the A's is assumed by A_n and as $A_n \leqslant 2j$ from (6), it follows from (11) that in terms of the a's, the relation (7) becomes

$$\left.\begin{aligned} 0 \;&\leqslant\; a_1 \;\leqslant 2j - (n-1) \,, \\ a_1 \;&\leqslant\; a_1 + a_2 \leqslant 2j - (n-1) \,, \\ a_1 + a_2 \;&\leqslant\; \sum_{i=1}^{3} a_i \leqslant 2j - (n-1) \,, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{i=1}^{n-1} a_i \;&\leqslant\; \sum_{i=1}^{n} a_i \leqslant 2j - (n-1) \,. \end{aligned}\right\} \tag{13}$$

The number N_M for any given M in (12), or for any given $\mu = J_{\max} - M$, subject to the relations (13) is given by the coefficient of x^μ in the product of the following $2j - (n-1)$ series [5]

$$\begin{aligned} &(1 + x + x^2 + \ldots + x^{b_1} + \ldots) \cdot \\ &\cdot (1 + x^2 + x^4 + \ldots + x^{2b_2} + \ldots) \cdot \ldots \cdot \\ &\cdot (1 + x^q + x^{2q} \ldots + x^{qb_q} + \ldots), \end{aligned} \tag{14}$$

subject to the condition that

$$\begin{aligned} &b_1 + b_2 + b_3 + \ldots + b_q \leqslant n \,, \\ &q = 2j - (n-1) \,. \end{aligned} \tag{15}$$

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[5] Cf. D. E. LITTLEWOOD: *The Theory of Group Characters* (Oxford, 1940), Chapter V.

TABLE I.

j	n	J
1/2	0, 2	0
	1	1
3/2	0, 4	0
	1, 3	3/2
	2	0, 2
5/2	0, 6	0
	1, 5	5/2
	2, 4	0, 2, 4
	3	3/2, 5/2, 9/2
7/2	0, 8	0
	1, 7	7/2
	2, 6	0, 2, 4, 6
	3, 5	3/2, 5/2, 7/2, 9/2, 11/2, 15/2
	4	0, 2(2), 4(2), 5, 6, 8
9/2	0, 10	0
	1, 9	9/2
	2, 8	0, 2, 4, 6, 8
	3, 7	3/2, 5/2, 7/2, 9/2(2), 11/2, 13/2, 15/2, 17/2, 21/2
	4, 6	2(2), 3, 4(2), 5(2), 6(3), 7, 8(2), 9, 10, 12
	5	1/2, 3/2, 5/2(2), 7/2(2), 9/2(3), 11/2(2), 13/2(2), 15/2(2), 17/2(2), 19/2, 21/2, 25/2
11/2	0, 12	0
	1, 11	11/2
	2, 10	0, 2, 4, 6, 8, 10
	3, 9	3/2, 5/2, 7/2, 9/2(2), 11/2(2), 13/2, 15/2(2), 17/2, 19/2, 21/2, 23/2, 27/2
	4, 8	0(2), 2(3), 3, 4(4), 5(2), 6(4), 7(2), 8(4), 9(2), 10(3) 11, 12(2), 13, 14, 16
	5, 7	1/2(2), 3/2(2), 5/2(2), 7/2(2), 9/2(6), 11/2(5), 13/2(4), 15/2(5), 17/2(4), 19/2(4), 21/2(3), 23/2(3), 25/2(2), 27/2(2), 29/2, 31/2, 35/2
	6	0(3), 2(4), 3(3), 4(6), 5(3), 6(7), 7(4), 8(6), 9(4), 10(5), 11(2), 12(4), 13(2), 14(2), 15, 16, 18

J values of states in equivalent-particle configuration $(j)^n$.
The number in the brackets following the value of J gives the number of appearances of that particular J value.

Finally, if we denote the number of levels having the value J by $\mathcal{N}_J$, it can be readily seen from the following arrangement

M	J_{max}	$J_{max}-1$	$J_{max}-2$	...
Number of states with M	N_0	N_1	N_2	...

that $\mathcal{N}_J$ is given by

$$\mathcal{N}_J = N_{J_{max}-J} - N_{J_{max}-(J+1)}. \tag{16}$$

A particular value of J will be absent if the right hand side of (16) is equal to zero. It also follows from (16) that there is always only one level with $J = J_{max}$ since $N_0 = 1$.

In the following table I, the values of J and the number of times each J appears are tabulated for all j up to 11/2 for all values of n allowed by the Pauli principle [6].

[6] Incidentally, it may be seen that a statement in MAYERS papers, *Phys. Rev.*, **78**, 20 (1950) that up to $j = 7/2$, each value of J appears only once, is not correct. For $j = 7/2$ and $n = 4$, the values $J = 2$ and 4 appear twice.

Un mesone τ uscente da stella.

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(ricevuto il 28 Marzo 1953)

In una coppia di lastre con emulsioni affacciate Ilford G5 di 1200 micron di spessore, esposta alla radiazione cosmica ad una quota di 30000 metri durante uno dei voli di pallone compiuti a Cagliari nell'estate 1952, è stato osservato un mesone τ uscente da una stella. La stella comprende (fig. 1), oltre al mesone τ stesso (indicato con M'), 10 rami al minimo, uno dei quali opposto al cono costituito dagli altri 9 può essere considerato il primario, e tre rami neri relativamente corti che terminano tutti nell'emulsione. Il primario e l'asse medio del cono delle particelle di sciame risultano inclinati di 60° rispetto al piano dell'emulsione, il che non consente la diretta determinazione della loro energia nè della loro natura; dal valore però dell'angolo mediano dello sciame e della sua molteplicità, si può stimare che l'energia del primario sia almeno di 30 GeV.

Il mesone τ esce all'indietro e forma un angolo di 58° col primario; dopo 5300 micron di percorso esce dalla prima emulsione del sandwich (lastra 76) e continua nella seconda (lastra 75) ove si arresta dopo altri 2000 micron di percorso (traccia M); decade quindi nel modo caratteristico dei τ dando origine a tre particelle la cui complanarità è confermata entro 2°.

Due di esse a e c dopo aver percorso rispettivamente 4100 e 1800 micron escono dalla parte del vetro, la terza b esce invece dalla parte dell'aria dopo 520 micron; gli angoli veri tra i tre rami risultano: $\widehat{ab} = 142°$, $\widehat{bc} = 103°$ e $\widehat{ca} = 115°$. Dalle misure dell'angolo medio di scattering sul ramo M e dalla relazione angolo di scattering-range si deduce per la massa della particella un valore di 890 ± 270 me. L'esatta reciproca posizione delle due lastre 75 e 76 durante il volo è stata stabilita tramite numerose traccie di primari pesanti con un'imprecisione non superiore a 50 micron; si è potuto stabilire così la corrispondenza delle due tracce M ed M' che è risultata biunivoca. Inoltre nella lastra 76 è stato pure trovato il prolungamento del secondario b uscente dalla disintegrazione del mesone τ; esso attraversa completamente l'emulsione 76 con un percorso di 2350 micron. Infine, dalle misure di conteggio dei grani e di angolo di scattering compiute sul ramo M', si è ricavato per tale ramo un valore della massa di 990 ± 150 me.

La confermata corrispondenza dei due rami M ed M' permette di considerarli come tracce di una unica particella e di determinare la massa di questa applicando le relazioni ionizzazione-range e scattering-range. I valori trovati in questo modo risultano rispettivamente di 935 ± 160 e 970 ± 116 me, con un valore medio risultante di 955 ± 110 me.

Una ulteriore determinazione della massa può venire compiuta tramite le particelle di decadimento. Sfortunatamente, causa lo sviluppo irregolare nello spessore dell'emulsione, non si sono potute ottenere misure di ionizzazione con la precisione che le lunghezze delle tracce avrebbero altrimenti consentito. Le determinazioni dell'angolo di scattering e della ionizzazione dei tre secondari (compreso il prolungamento di b) forniscono per l'energia dei secondari, ammesso che tutti e tre siano mesoni π, i seguenti valori: $E_a = 30{,}3 \pm 2{,}3$, $E_b = 30 \pm 4$, $E_c = 17{,}9 \pm 1{,}3$ MeV; ciò fornisce per l'energia di decadimento un valore di $Q = 78 \pm 5$ MeV e un valore della massa del τ di 975 ± 12 me. Applicando il bilancio della conservazione della quantità di moto, i valori dell'energia di decadimento che si possono ottenere basandosi successivamente sui tre valori sperimentali dell'energia dei rami a, b e c sono i seguenti: $Q_a = 70{,}3$, $Q_b = 79{,}5$, $Q_c = 94{,}4$, la cui media dà $Q = 81{,}4$ MeV.

La conservazione della quantità di moto verrebbe soddisfatta entro gli stessi limiti di approssimazione assumendo che il ramo c anzichè un mesone π fosse un mesone μ. L'evidenza contraria a tale ipotesi, ottenuta finora dall'analisi accurata di altri mesoni τ [1], [2] [3] rende però piuttosto improbabile questa seconda eventualità.

L'evento è stato trovato dal sig. A. BERNARDI che ha pure collaborato alle misure.

Ci è grato cogliere questa occasione per ringraziare tutti coloro che hanno collaborato nel lancio dei palloni della spedizione scientifica in Sardegna ed in modo particolare la Marina e l'Aeronautica Militare Italiana per i mezzi messi a disposizione e per l'incondizionato aiuto fornitoci.

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[1] M. CECCARELLI, N. DALLAPORTA, M. MERLIN e A. ROSTAGNI: *Rend. Acc. Naz. Lincei*, s. VIII, **13**, 9 (1952); *Nature*, **170**, 454 (1952).

[2] Convegno di Londra del 29 Gennaio 1953.

[3] Comunicazione privata del gruppo di Milano (M. PANETTI e L. SCARSI).

Fig. 1.

Sulla penetrazione degli elettroni.

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(ricevuto il 28 Marzo 1953)

La probabilità differenziale $\pi(E_0, E, t)$ che un elettrone di energia iniziale E_0 possieda, dopo aver attraversato t unità di radiazione, l'energia compresa fra E e $E + \mathrm{d}E$, è data dalla soluzione dell'equazione

$$(1) \qquad \frac{\partial \pi(E,t)}{\partial t} = -\int_0^1 \left[\pi(E,t) - \frac{1}{1-v}\,\pi\left(\frac{E}{1-v}, t\right)\right]\varphi(v)\,\mathrm{d}v + \beta\,\frac{\partial \pi(E,t)}{\partial E},$$

con la condizione iniziale $\pi(E, 0) = \delta(E - E_0)$. La (1) si ottiene dall'equazione di diffusione per la componente elettronica della teoria della cascata [1], ignorando a secondo membro il contributo dovuto alla materializzazione dei fotoni creati, e determina pertanto l'andamento di tale componente per le prime unità di radiazione. Con la precedente condizione iniziale, la funzione $\pi(E, t)$ acquista il significato di probabilità nel senso già precisato. La funzione $\varphi(v)$ è la sezione d'urto differenziale per irraggiamento, ed il primo termine a secondo membro tiene conto dello « straggling » dovuto a tale processo, mentre il secondo termine, dove β è la perdita costante di energia per unità di radiazione, dà il contributo dovuto al processo di ionizzazione.

Trascurando il termine in β e facendo ricorso all'artificio della trasformata di Mellin, la soluzione della (1), come è stato mostrato da Eyges [2], è immediata. Dallo stesso autore è stata data anche una soluzione della (1) col termine di ionizzazione, sotto forma di uno sviluppo in serie di potenze in t, il quale possiede come primo termine la precedente soluzione, salvo la sostituzione di E_0 con $E_0 - \beta t$. La validità di tale sviluppo è però limitata ad energie non molto diverse dal valore iniziale E_0, in quanto esso diverge per E tendente a zero. Qui mostreremo come della (1) si possa dare, evitando lo sviluppo in serie, una soluzione che semplici applicazioni mostrano essere un'ottima approssimazione di quella esatta, e che è

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(1) B. Rossi e K. Greisen: *Rev. Mod. Phys.*, **13**, 240 (1941).
(2) L. Eyges: *Phys. Rev.*, **76**, 264 (1949).

inoltre valida anche per $E = 0$. Trascurando infatti la quantità $v\beta t$ rispetto ad $E + \beta t$, il che appare lecito osservando l'andamento della $\varphi(v)$, si vede per sostituzione diretta che la funzione

$$\pi(E_0, E, t)\,\mathrm{d}E = \frac{\mathrm{d}E}{E_0}\,\frac{1}{2\pi i}\int_{\delta - i\infty}^{\delta + i\infty} \mathrm{d}s \left(\frac{E_0}{E + \beta t}\right)^{s+1} \exp\left[-A(s)t\right], \tag{2}$$

avendo posto

$$A(s) = \int_0^1 [1 - (1 - v)^s]\varphi(v)\,\mathrm{d}v\,, \tag{3}$$

è la soluzione della (1) che soddisfa alla condizione iniziale data. Dalla (2) segue l'espressione della probabilità integrale

$$\Pi(E_0, E, t) = \int_E^\infty \pi(E_0, E, t)\,\mathrm{d}E = \frac{1}{2\pi i}\int_{\delta - i\infty}^{\delta + i\infty} \mathrm{d}s \left(\frac{E_0}{E + \beta t}\right)^{s} \exp\left[-A(s)t\right]. \tag{4}$$

Per $\beta = 0$ le espressioni date si riducono a quelle calcolate da Eyges tenendo conto del solo processo di irraggiamento.

Le integrazioni in campo complesso che figurano nelle (2) e (4) si possono effettuare col metodo dei residui usando per $\varphi(v)$ ed $A(s)$ le espressioni approssimate

$$\varphi(v) = -b\,\frac{(1 - v)^a}{\ln(1 - v)}\,, \qquad A(s) = b \ln\left[\frac{1 + a + s}{a + s}\right]. \tag{5}$$

Con tale scelta, la (2) presenta un polo di ordine bt per $s = -(1 + a)$, e così pure la (4), che possiede inoltre un polo semplice per $s = 0$. Con $y = \ln\left(E_0/(E + \beta t)\right)$, le rispettive soluzioni sono

$$\pi(E_0, E, t)\,\mathrm{d}E = \frac{(1 + a)^{bt}}{\Gamma(bt)}\left(\frac{E + \beta t}{E_0}\right)^a \ln\left(\frac{E_0}{E + \beta t}\right)^{bt-1}\frac{\mathrm{d}E}{E_0}\,, \tag{6}$$

$$\Pi(E_0, E, t) = 1 - \exp\left[-(1 + a)y\right]\sum_{n=0}^{bt-1}(1 + a)^n\,\frac{y^n}{n!}\,. \tag{7}$$

Per $\beta = 0$, $a = 0$ e $b = 1$, la (6) si riduce alla funzione calcolata da Bethe e Heitler [3]; per $a = 0$ e $b = 1$, la (7) non è altro che la rappresentazione per valori interi di t della funzione

$$\Pi(E_0, E, t) = \frac{1}{\Gamma(t)}\int_0^y e^{-x}x^{t-1}\,\mathrm{d}x\,, \tag{8}$$

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[3] H. Bethe e W. Heitler: *Proc. Roy. Soc.*, **146**, 83 (1934).

la quale, a parte il diverso significato della variabile y, è la nota espressione della probabilità integrale (4).

Per $t = 1$, la sommatoria su n si riduce al primo termine e la (7) si scrive

$$\Pi(E_0, E, 1) = 1 - \exp[-y] = 1 - \frac{E+\beta}{E_0}. \tag{9}$$

Per $E_0 = 2\beta$, ad esempio, la (9) si annulla, come dev'essere, per $E = \beta$, ed al limite si ha $\Pi(2\beta, 0, 1) = 0{,}5$. Il primo termine dello sviluppo in serie di Eyges, usando sempre le espressioni approssimate (5), sarebbe dato per $t = 1$ dalla

$$\Pi_1(E_0, E, 1) = 1 - \frac{E}{E_0 - \beta}. \tag{10}$$

Anche la (10), con $E_0 = 2\beta$, si annulla per $E = \beta$, ma tende al valore uno per $E \to 0$, e solo tenendo conto dei successivi termini si può correggere l'andamento previsto dalla (10), almeno fino ad un certo valore minimo di E, non essendo possibile dare, per l'accennata divergenza, il valore asintotico. Il contributo di questi termini per il caso particolare considerato di $E_0 = 2\beta$ e $t = 1$ è rappresentato nella fig. 4 del lavoro di Eyges; pur essendo stata usata un'approssimazione diversa da quella che corrisponde alle (5), la figura illustra tuttavia la differenza fra la soluzione data da Eyges e quella espressa dalla (9), o in generale dalla (4).

Con una scelta diversa per la $\varphi(v)$, per valutare gli integrali (2) e (4) bisogna ricorrere al metodo del punto di sella. In tal modo si trova per la probabilità differenziale

$$\pi(E_0, E, t)\,\mathrm{d}E = \frac{1}{\sqrt{2\pi}} \frac{\exp[-A(s_0)t]}{\{-A''(s_0)t\}^{1/2}} \left(\frac{E_0}{E+\beta t}\right)^{s_0+1} \frac{\mathrm{d}E}{E_0}, \tag{11}$$

dove il punto di sella s_0 e la variabile t sono legati dalla relazione

$$\ln \frac{E_0}{E+\beta t} = A'(s_0)t. \tag{12}$$

In modo analogo per la probabilità integrale si ottiene

$$\Pi(E_0, E, t) = \frac{1}{\sqrt{2\pi}} \frac{\exp[-A(s_0)t]}{\{-A''(s_0)t + (1/s_0)^2\}^{1/2}} \left(\frac{E_0}{E+\beta t}\right)^{s_0}, \tag{13}$$

con la condizione:

$$\ln \frac{E_0}{E+\beta t} = A'(s_0)t + \frac{1}{s_0}. \tag{14}$$

Usando la stessa approssimazione di Eyges, le (13) e (14) danno $\Pi(2\beta, 0, 1) = 0{,}35$; arrestando lo sviluppo ai termini in t^4, per $E = 0{,}3\beta$ Eyges ottiene $\Pi(2\beta, 0.3\beta, 1) = 0{,}4$.

(4) H. J. Bhabha e W. Heitler: *Proc. Roy. Soc.*, **159**, 432 (1937).

Current Fluctuations in Corona Counters.

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(ricevuto il 13 Aprile 1953)

In a previous work [1] the behaviour of cylindrical counters at the zone of corona discharge has been studied and the possibility of their use as proportional counters having a multiplication factor independent of voltage has been established.

It has also been observed that the average current i_0 was accompanied by a roughly sinusoidal component.

Since then, a theoretical analysis of the dynamical behaviour of the corona current has been attempted and the following equation was obtained:

$$\frac{1}{i}\frac{\mathrm{d}i}{\mathrm{d}t} \quad \frac{\alpha}{\tau}\left(V - V_s - \frac{1}{4\pi\varepsilon_0}\int_0^T i(t-\theta)f(\theta)\,\mathrm{d}\theta\right),$$

in which

i = corona current,
V = voltage applied at the counter,
V_s = voltage of corona treshold,
T = transit time of ions,
τ = transit time of electrons,
$\eta = \left(\frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}V}\right)_{V=V_s}$,
N = multiplication factor of the counter operated in the conventional proportional zone,
$f(\theta)$ = weight function.

In the above equation the fact was taken into account that the multiplication factor, and hence the percentage value of the time rate of change of current, depends upon the electric field at the wire which in turn is determined by the voltage applied at the counter and by the space charge distribution there. This latter depends upon a weighed average of the preceding values of current.

A stable solution of the equation is a costant i_0. Every impressed disturbance generates damped oscillations of which the period and damping has been calculated.

The observed permanent oscillation is attributed to the excitation of a resonant frequency of the counter by the continuous noise spectrum of the photoelectronic current from the cathode which is responsible for the selfsustenance of the corona discharge.

The observed effect is unusually great, because the photoelectronic current is very small compared with the corona current due to the high multiplication factor (about 1 000).

The theory agrees with the experimental data of the work quoted above, which, however, are not by themselves sufficient as complete proof; but experiments are now in progress to such end.

[1] L. Colli, U. Facchini e E. Gatti: *Rev. Scient. Inst.*, **23**, 621 (1952).

Analysis of a τ-Meson Decay.

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(ricevuto il 14 Aprile 1953)

1. – An analysis is given of a τ-meson ([1-8]), which decays into three π-mesons; one of the π-mesons undergoes the typical transmutation $\pi^+ \to \mu^+ \to e^+$ ([8]).

The event has been observed in an Ilford G5 nuclear plate 600 microns thick. The plate was exposed for six hours in a thin aluminium container at the altitude of about 25000 m above sea level, 40° North geomagnetic latitude. The plates were exposed in the European Balloon flights in Sardinia during Summer 1952.

The event is shown in the microphotograph. The parent particle stops in the emulsion after a path of 4 mm at an angle of about 100° from the mean upward vertical direction in the plate.

The fast particles (*a*) and (*b*) travel in nearly opposite directions. The slow particle (*c*) ends in the emulsion after 162 microns giving rise to a secondary particle which comes to rest in the emulsion after 560 microns and decays in turn into an electron of $p\beta = (38 \pm 11)$ MeV/*c*. We can therefore identify (*c*) as a slow π^+-meson.

Owing to the strong asymmetry in the present example of decay, only the projected angle (*ab*) may be determined with great accuracy since it is defined by

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([1]) R. Brown, U. Camerini, P. H. Fowler, H. Muirhead, C. F. Powell and D. M. Ritson: *Nature*, **163**, 82 (1949).

([2]) J. B. Harding: *Phil. Mag.*, **41**, 405 (1950).

([3]) P. H. Fowler, M. G. K. Menon, C. F. Powell and A. Rochat: *Phil. Mag.*, **42**, 1040 (1951).

([4]) P. E. Hodgson: *Phil. Mag.*, **42**, 1060 (1951).

([5]) M. Ceccarelli, M. Dallaporta, M. Merlin and A. Rostagni: *Nature*, **170**, 454 (1952).

([6]) M. Ceccarelli, M. Dallaporta, M. Merlin, G. Quareni and G. T. Zorn: Private communication (1952).

([7]) A. J. Herz, P. E. Hodgson and R. M. Tennent: *Phil. Mag.*, **44**, 85 (1953).

([8]) Roma group: Discussion of the Royal Society (London, January 1953).

two straight tracks. The errors on all the real angles are about 1° (*); they have been calculated taking into account: 1) the errors on the projected angles, 2) the errors on the dip angles, which are partly due to uncertainty of the shrinkage factor. In fact the determination of the angles is affected by a systematic error due to the uncertainty in the original value of the plate thickness and to the difficulty of taking into account its variations in different conditions of moisture content.

Coplanarity of a, b, c is satisfied within 1° 40′.

The geometrical data are summarized in Table I.

TABLE I.

Particle	Lenght (microns)	Dip angle	Angle between the tracks	
			projected	real (calculated)
τ	4 000	—	—	—
a	2 550	+ 11° 40′	$(ab) = 171° 48'$	$(a'b') = 170° 40' = \lambda$
b	640	— 7° 10′	—	$(b'c') = 43° 10' = \eta$
c	162	— 4°	$(ca) = 145° 02'$	$(c'a') = 144° 30' = \theta$

2. – Experimental measurements have been carried out to determine the mass of the heavy meson.

Scattering-range measurements on the track of the parent particle give:

$$m_\tau = (890 \pm 180)\ m_e \quad (**).$$

It is possible to obtain a more accurate result by the method of the energy balance, i.e. by determining the sum of the rest and kinetic energies of the decay products, assuming that the conservation of the momentum holds and that no neutral particle is ejected.

For this purpose, grain density and scattering measurements on the tracks (a) and (b) have been made and give the following results:

$$g_a = 1{,}72 \pm 0{,}1 \ ; \qquad p_a\beta_a = (84 \pm 12)\ \text{MeV/c} \ ;$$

$$g_b = 1{,}95 \pm 0{,}15 \ ; \qquad p_b\beta_b = (72 \pm 15)\ \text{MeV/c} \ .$$

We obtain the following values for the masses:

$$m_a = (270 \pm 43)\ m_e \ ; \qquad m_b = (284 \pm 63)\ m_e \ .$$

Assuming that both particles are π-mesons, the momenta and the kinetic

(*) The values of the real angles depend of course on the mutual position of the tracks in the space.

(**) Standard deviations are given here and in the following.

Fig. 1.

Observed by P. Formenti

energies can be deduced from the ionization and, with lower accuracy, from the scattering measurements (Table II).

TABLE II.

Particle	p (MeV/c) (from ionization)	E (MeV) (from ionization)	p (MeV/c) (from scattering)	E (MeV) (from scattering)
a	127 ± 5	$48{,}5 \pm 3{,}5$	126 ± 10	$48 \pm 6{,}8$
b	$110{,}5 \pm 9$	$38{,}5 \pm 5{,}5$	113 ± 13	$40 \pm 8{,}5$

The nature of the slow particle (c) is clearly shown by its decay; its energy is: $E_c = (2{,}2 \pm 0{,}05)$ MeV with $p_c = (25{,}1 \pm 0{,}3)$ MeV/c.

The Q of the disintegration and the mass of the τ-meson can be deduced from the above measurements. We obtain:

$$Q = (89{,}7 \pm 9) \text{ MeV} ; \qquad m_\tau = (1\,006 \pm 21)\ m_e . \quad (*)$$

A better determination of Q and m_τ can be obtained, in our case, using the values of the momenta p_a, p_b, p_c and of the angles λ, η, θ (Table I) between the tracks considering that the accuracy of momentum p_c is largely the best, and the errors on the real angles are nearly equal.

We made a graphical determination instead of an analitical one in order to take into account more easily the strong sensitivity of p_a and p_b to small variations of the angle λ. We plot the three families of lines: $p_a = p_c \sin \eta / \sin \lambda$; $p = -p_c \sin(\eta + \lambda)/\sin \lambda$; $\theta = 360^\circ - (\eta + \lambda)$, versus λ, taking the angle η as a parameter. Considerations on the experimental errors suggest that we should choose the value $\eta = 43^\circ\,40'$; λ is obtained through weighted means, the other values resulting so determined. The energy is calculated for the three tracks, hence the Q of the decay $\tau \to \pi^+ + \pi^- + \pi^\pm$ and the mass of the τ-meson are evaluated, their error including the uncertainty of the π-meson mass.

Table III shows the results of the energy balance.

TABLE III.

Particle	Momentum (MeV/c)	Kinetic energy (MeV)
a	$117{,}3 \pm 5{,}1$	$42{,}4 \pm 3{,}5$
b	$97{,}8 \pm 9{,}3$	$30{,}6 \pm 5{,}6$
c	$25{,}1 \pm 0{,}3$	$2{,}2 \pm 0{,}05$

$$Q = (75{,}2 \pm 7) \text{ MeV} ; \qquad m_\tau = 977{,}7 \pm 17)\ m_e .$$

(*) We adopt the values of the π-mass, as determined by BARKAS *et al.*: *Phys. Rev.*, **82**, 102 (1951): $m_{\pi^+} = (277{,}4 \pm 1{,}1)\ m_e$; $m_{\pi^-} = (276{,}1 \pm 1{,}3)\ m_e$.

3. – The frequency of occurrence of slow heavy mesons compared with that of slow π-mesons is briefly summarized in the following.

One τ-meson, two K-mesons (of which one is ejected from star [9]), 360 $\pi^{\pm}$ (of which 30 are ejected from stars) together with 3 900 stars of three or more tracks have been recorded in our laboratory on 11,7 cm^3 of Ilford G5 emulsion. The plates, which are coated with 600-1 200 microns emulsion, have been exposed in various flights between 20 000 and 25 500 m above sea level and were not surrounded with condensed matter.

(9) R. LEVI SETTI and G. TOMASINI: *Nuovo Cimento*, **9**, 1244 (1952).

LIBRI RICEVUTI E RECENSIONI

LEONARDO DA VINCI – *I Libri del volo* nella ricostruzione critica di ARTURO UCCELLI, con la collaborazione di CARLO ZAMMATTIO. Editore Ulrico Hoepli, Milano 1952. Un vol. in-8° di pp. XXXIV+238, Lire 6000.

Nell'introduzione analitica a questa attesa e bella edizione dei *Libri del volo* di LEONARDO, gli AA. avvertono di aver preferito seguire, nel riordino del materiale dei vari Codici, anzichè il metodo cronologico il metodo « logico » già da loro adottato per i *Libri di meccanica*, pubblicati da Hoepli nel 1940-42. Sulla base delle indicazioni lasciate dallo stesso LEONARDO, gli appunti raggruppabili sotto i titoli « del moto dei venti », « delle cose che discendono infra l'aria » e « della resistenza dell'aria » precedono quindi il materiale del *Codice sul volo degli uccelli* e degli altri testi leonardeschi dedicati al volo animale. Analogamente, i sei « libri » in cui è stata suddivisa questa materia precedono quello, « del volo strumentale » ovvero meccanico, che LEONARDO voleva apparisse a conclusione del suo Trattato, pur contenendo proprio gli studi di macchine per il volo dai quali ebbe inizio il suo interesse per gli altri argomenti del Trattato stesso.

Non si può che approvare questo rispetto delle sue intenzioni da parte degli autori della presente raccolta, tanto più se si considera che l'unica finora apparsa, quella curata nel 1936 da R. GIACOMELLI, già osserva nelle grandi linee il criterio di un ordinamento storico-cronologico. Seguendo invece il metodo logico viene a mancare, naturalmente, l'equa ripartizione del materiale tra i vari « libri », dato che su alcuni argomenti del suo programma LEONARDO risulta aver lasciato un numero molto minore di appunti che su altri; ma la cosa ha un'importanza relativa, come non disturba il fatto che gli strumenti meteorologici siano dovuti finire nell'ampia introduzione alla raccolta. Dispiace semmai, data l'eleganza anche tipografica della pubblicazione e la ricchezza di note bibliografiche scrupolose, notare la presenza qua e là di errori di stampa facilmente evitabili. Gli oltre 400 disegni sono in genere assai chiari, ed appare utile ai fini di uno studio dei dettagli tecnici, spesso opportunamente ingranditi, il rinforzo o completamento dato al tratto originale.

Non è il caso di ricordare qui con esempi il molteplice fascino che si sprigiona dalle pagine di LEONARDO, fascino dovuto alla frequenza delle intuizioni precorritrici, in particolare nei frammenti che si sovrappongono a quelli dei *Libri di meccanica* trattando soggetti di interesse fisico generale, alla acutezza delle innumerevoli osservazioni dirette sull'anatomia e sulle manovre degli uccelli nelle più diverse condizioni di volo, all'ingegnosità dei vari congegni invano escogitati per la realizzazione del sogno del volo umano muscolare. Basti osservare come anche da questa paziente e minuziosa ricostruzione del volume progettato da LEONARDO, la quale comprende disegni e appunti sparsi non considerati nella precedente o reinterpretati, appaia chiara l'evoluzione del pensiero di lui verso un sistema atto ad imitare il volo librato e veleggiato dei grandi rapaci, piuttosto che il volo battente degli uccelli minori e degli insetti. Dai complessi meccanismi per la battuta delle ali, si passa infatti a quello che è stato chiamato l'« aliante » di LEONARDO: una coppia di ali aventi la struttura di quelle dei

pipistrelli e dimensionate in modo da sostenere, in condizioni di vento relativo, l'uomo sospeso verticalmente in mezzo per abbassare il baricentro e poter modificare con movimenti del corpo le condizioni di equilibrio dell'insieme. Il sistema non differisce sostanzialmente da quello adottato da Otto Lilienthal per le migliaia di esperimenti di volo librato con cui, a fine '800, egli fondò la scienza pratica del volo col « più pesante dell'aria »; costituisce quindi un titolo, a vantaggio di LEONARDO, ben maggiore di quello attribuitogli dai ripetuti ma superficiali paragoni tra i suoi sistemi ad ali battenti e l'aeroplano ad elica dei fratelli Wright.

I disegni leonardeschi relativi a questo progetto di aliante sono più di uno, ed in tutti sembra evidente il nesso tra la posizione sospesa del volatore e la manovra separata delle ali, ridotta ad una flessione o svergolatura delle metà esterne di esse; se si considera la mole di progetti dedicata da LEONARDO al problema di ridurre mediante ruotismi di demoltiplicazione lo sforzo muscolare dell'uomo, riesce difficile pensare ad un suo ritorno alla concezione, tipica delle raffigurazioni artistiche del mito di Dedalo e Icaro, di una azione diretta di battuta simultanea delle ali da parte delle braccia. A. UCCELLI e C. ZAMMATTIO appaiono invece condividere in proposito l'opinione del GIACOMELLI, il quale attribuisce caratteristiche di ornitottero semplice a buona parte di tali progetti, e d'altro lato trascura proprio l'interessantissimo disegno del Codice Atlantico (folio 22, verso *b*) su cui questi autori si soffermano per riconoscere in esso un progetto di aliante ad ali mobili.

Dove gli autori dissentono dal GIACOMELLI, è riguardo la dibattuta questione del principio di azione e reazione; sulla traccia di ROBERTO MARCOLONGO, essi ritengono possa vedersi un'enunciazione anticipata del terzo principio della dinamica in varie frasi di LEONARDO che ripetono il concetto delle seguenti:

« Perchè tanto è a movere l'aria contro all'uccello per se immobile, quanto movere l'uccello contro all'aria quieta »,

« Tanto è a movere la strada sotto il peso, quanto il peso sopra la strada ».

Che questo concetto di reciprocità, riesposto poi da NEWTON tra i corollari dei *Principia*, sia importante per l'applicazione del metodo dei modelli, chiaramente preconizzata anche altrove da LEONARDO, è cosa che nessuno pone in dubbio, a parte ogni questione storica, pericoloso è invece l'equivoco, come si vede non infrequente, tra la simmetria del tipo di frasi citate e quella del III principio, nel quale entra appunto la coppia di concetti azione-reazione, che non ha nulla a che fare con l'idea di movimento relativo.

VITTORIO SOMENZI

S. FLÜGGE und H. MARSCHALL – *Rechenmethoden der Quantentheorie*. 1. Teil, 2. Auflage, S. 272 + VII Springer–Verlag - Berlin. Göttingen. Heidelberg.

Il libro di FLÜGGE e MARSCHALL, che è ora alla sua seconda edizione, a cinque anni di distanza dalla prima, costituisce ciò che propriamente noi diremmo un testo di Esercizi e Problemi di Meccanica Quantistica. Il presente volume, che tratta esclusivamente delle applicazioni del metodo di Schrödinger, dovrebbe essere seguito, secondo le intenzioni e le promesse degli autori, da un secondo volume del quale formeranno principale oggetto le applicazioni del metodo di Dirac; verrà così coperto tutto il campo, di interesse didattico, della Meccanica quantistica.

Questa seconda edizione della prima parte, rappresenta non solo un miglioramento rispetto alla prima edizione per quei problemi che le due edizioni hanno in comune, ma anche un ampliamento, stante il fatto che il numero dei problemi

è salito da 79 a 106. I problemi sono preceduti da una brevissima introduzione in cui vengono riassunti i punti essenziali del metodo di Schrödinger, allo scopo, soprattutto, di richiamare il necessario formalismo. Il complesso è poi diviso in Problemi a un corpo (i primi 80) e Problemi a due o più corpi; parte risolti esattamente, come gli ormai classici casi con barriere o buche di potenziale fatte a gradini, con potenziale parabolico, coulombiano, ecc.; parte risolti con metodi approssimati. Tra questi ultimi particolarmente figurano i problemi risolti col metodo di W.K.B. ma non mancano applicazioni del più recente, ma ormai classico metodo variazionale di Schwinger per il calcolo delle fasi.

I problemi d'urto occupano una parte rilevante del volume, in coerenza con la importanza che essi hanno assunto nella letteratura scientifica; molto opportuna sembra anche una ampia discussione contenuta nei problemi a più corpi, delle proprietà statistiche dei sistemi di particelle ed in particolare del gas di Fermi.

In conclusione il volume di Flügge e Marschall, scritto con molta cura e serietà di intenti, può costituire un prezioso ausilio per l'insegnante di Meccanica quantistica, per gli studenti e in generale per chiunque lavori in fisica. La scelta di un buon libro di Esercizi e Problemi è infatti altrettanto importante della scelta di un buon libro di testo.

G. Puppi

Grimsehl – *Lehrbuch der Physik*, III Band: *Optik*. 12ª edizione; B. G. Teubner, Leipzig, 1952.

Il classico libro di Grimsehl viene pubblicato attualmente suddiviso in un numero di volumi maggiore di quello originario. Il terzo volume che ora compare con la collaborazione di Seeliger e sotto la cura di Schallreuter comprende l'Ottica. In esso sono naturalmente trattati gli argomenti che riguardano l'ottica ondulatoria e l'ottica della radiazione, non limitatamente alle sole radiazioni visibili.

Le qualità del libro di Grimsehl che si sono sempre più affermate nelle successive edizioni, sono ben note, e cioè i vari argomenti sono trattati da un punto di vista estremamente chiaro e semplice; esiste in tutta la esposizione una grande aderenza ai fatti sperimentali, e la presenza di numerose figure chiare e significative rende la lettura molto piana. Il lettore che abbia solo le conoscenze fondamentali della scuola media classica o scientifica ha modo di penetrare nel vivo degli argomenti che sono esposti nella loro parte essenziale.

Le difficoltà che si presentano nell'esposizione di una parte della Fisica così estesa come l'Ottica, oltre alla necessità di una grande chiarezza dei vari concetti, consiste in una scelta oculata degli argomenti che, per praticità ed utilità, devono essere necessariamente limitati. La nuova edizione di questo libro si presenta accuratamente aggiornata rispetto a tutti gli argomenti moderni. L'esposizione segue la suddivisione ormai classica, e cioè dopo aver parlato delle leggi della propagazione della radiazione, e della misura dell'intensità luminosa, passa a trattare i fenomeni della riflessione e della rifrazione e poi, in modo sintetico ma soddisfacente, i fenomeni dell'interferenza, della diffrazione e polarizzazione. Dopo aver studiato tutte le possibili manifestazioni luminose, si passa allo studio degli apparecchi che possono essere meglio compresi nel loro funzionamento. Nella parte finale si parla della genesi e dello assorbimento delle radiazioni, in modo da fornire la base sperimentale dei fenomeni da cui è originata la teoria quantistica. Non manca la parte che riguarda i fenomeni della relatività la cui lettura è molto consigliabile, perchè in questa parte — ed

in modo convincente — è trattata la diversità fra gli effetti del primo e del secondo ordine e la impossibilità della dimostrazione sperimentale del moto assoluto. Chiude il volume un capitolo sull'Ottica fisiologica. Come negli altri volumi e come nelle precedenti edizioni non manca qualche limitato, ma fondamentale accenno storico e qualche etimologia, cioè nozioni di cultura che non guastano e che permettono di inquadrare meglio l'esposizione.

A. CARRELLI

E. T. JAYNES: – *Ferroelectricity*. – Pag. VIII + 137. Princeton University Press – Princeton, 1953.

Il volume, primo di una collana di « Investigations in Physics », iniziata sotto la direzione di E. WIGNER ed R. HOFSTADTER, tratta di un argomento assurto, negli ultimi anni, a grande importanza sia dal punto di vista scientifico, sia da quello delle applicazioni, come è provato dalla copiosa fioritura di lavori originali comparsi nelle Riviste, o presentati in recenti congressi.

Della *Ferroelettricità*, nelle sue diverse manifestazioni che richiamano alla mente il ferromagnetismo, si può trovare traccia in qualche trattato di Piezoelettricità, ad esempio, in quello del CADY: *Piezoelectricity* (Mc Graw-Hill, 1946); una trattazione specifica completa della ferroelettricità non è pero, a quanto mi consta, mai stata fatta.

Il volume del JAYNES, pur nella sua piccola mole, riesce perciò maggiormente interessante per la visione sintetica che esso riesce a dare dell'argomento, e dei problemi più importanti che attualmente si agitano in esso.

Un primo capitolo è dedicato al richiamo dei fatti sperimentali fondamentali, inizialmente riscontrati nel Sale di Rochelle. Dopo un richiamo della interpretazione di essi, fatta sulla base della esistenza di un campo interno (tipo campo molecolare di Weiss) vengono additate le principali difficoltà per una soddisfacente teoria, prima fra tutte l'esistenza di due punti di Curie, che d'altra parte sembra logico giustificare con uno stesso meccanismo.

Vengono quindi ricordate le caratteristiche di altri due tipi di ferroelettrici, dando molto rilievo a quelle del titanato di bario, $BaTiO_3$.

In un secondo capitolo sono esposte le teorie sulla ferroelettricità del $BaTiO_3$. Nella ipotesi di uno spostamento di ioni in seno ai particolari reticoli cristallini che caratterizzano questo composto (cristallo ionico) sono state sviluppate teorie da parte di MASON e MATTHIAS, e di DEVONSHIRE e SLATER. Una teoria fondata sullo spostamento degli ioni ossigeno, secondo alcuni preponderante su quello degli altri ioni, viene pure ricordata. Infine, con maggiore estensione viene esposta una teoria « elettronica » sviluppata essenzialmente dall'Autore: a favore della possibile validità di essa depongono alcuni fatti sperimentali acquisiti, ma esperienze più probative appaiono desiderabili.

Alcuni aspetti del problema della ferroelettricità sono quindi trattati da un punto di vista termodinamico: tra questi l'anomalia del calore specifico e taluni complementi alla teoria di Devonshire.

L'ultima parte del volume è dedicata a due importanti questioni: quella del campo interno nei cristalli, e quella della loro polarizzazione elettrica.

Nelle calcolazioni inerenti ai cristalli ionici si trascurano spesso, a scopo di semplificazione, certe speciali situazioni che in questi corpi possono invece assumere carattere determinante. Così, per esempio, un determinato ione, specialmente se di segno negativo, come è il caso dell'ossigeno, deve mostrare differenti polarizzabilità in relazione con il

carattere ionico dei cristalli (appartenenti a diversi composti chimici) in cui esso si trova.

E d'altra parte il campo interno viene sempre calcolato come se si trattasse di dipoli e fosse uniforme, mentre invece spesso si tratta di campi di multipoli, che in prossimità degli ioni ne alterano i momenti e la polarizzazione. Per questo l'A., ricordati taluni metodi per la valutazione del campo in determinati punti di un reticolo (EWALD), sviluppa alcuni sistemi di calcolo, già oggetto di sue originali ricerche, che appaiono più elementari di quelli di EWALD. Questi metodi possono servire al calcolo dei campi macro- e microscopici, come pure a quello della polarizzazione spontanea ed indotta.

Nel complesso il volumetto di JOYNES, per la sua struttura attuale e con la sua copiosa ed aggiornata bibliografia può, più che altro, costituire una introduzione generale allo studio della ferroelettricità, per il quale fornisce allo studioso anche una rivista delle più accreditate teorie, ma naturalmente non di tutte. E così dicasi di molta parte degli argomenti particolari, che non trovano posto nel lavoro ma si possono ripescare nella bibliografia, aggiornata a tutto il 1951.

ANGELO DRIGO

An International Bibliography on Atomic Energy. Vol. 2, Scientific Aspects. Supplement N. 1. Atomic Energy Section Department of Security Council Affairs. United Nations, New York, 1952.

Questo primo supplemento di aggiornamento al grosso volume pubblicato dall'O.N.U. nel 1951 contiene la bibliografia mondiale dei lavori pubblicati negli anni 1949 e 1950 nei seguenti campi: fisica nucleare fondamentale; fisica e ingegneria dei reattori nucleari; effetti medici e biologici delle radiazioni di grande energia; gli isotopi nella biologia e nella medicina; applicazioni degli indicatori radioattivi nelle scienze non biologiche e nella tecnologia.

Come avverte S. A. KORFF nella prefazione, in questo volume si sono apportati alcuni miglioramenti rispetto al precedente, sia nella compilazione dell'indice, sia nei criteri di ordinamento del materiale nelle diverse sezioni, curando nuove suddivisioni che semplificano notevolmente la ricerca bibliografica.

Il volume si presenta in eccellente veste tipografica ed è assai curato nella composizione: basti dire che perfino i titoli dei lavori italiani sono riprodotti senza errori.

C. SALVETTI

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